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The boundary support of shells

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THE BOUNDARY SUPPORT OF SHELLS

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ABSTRACT

This thesis studies the conditions under which a shell can carry loads by membrane action only. In order for a shell to work in this way it must be impossible for the shell to deform without changing lengths on the surface.

The problem is studied using statics and kinematics and it is found that the behaviour is governed both by the geometry of the shell and its boundary supports.

to all my family

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NOTATION

Although all symbols are defined when first introduced, the following is a list of frequently used symbols.

x^i, ϑ^i	Systems of Cartesian, and curvilinear coordinates respectively
$r(\vartheta^\alpha, t)$, $r(\vartheta^\alpha)$	The surface position vector in the present and the reference configuration
$\vartheta^\alpha, \vartheta^3$	The curvilinear coordinates along and through the surface
a^α, a_α	The contravariant and covariant base vectors
a_3, n	The unit normal to the surface
ds_α, ds_α	The line element vector and its magnitude
ds^2	The line element of the surface
dS	Element of area
$d\tau$	Element of volume
$a_{\alpha\beta}, a^{\alpha\beta}$	The covariant and contravariant metrics of the surface
a	The determinant of the first fundamental form
δ^α_β	The mixed second order tensor : Kronecker delta
$b_{\alpha\beta}, b^{\alpha\beta}, b^\alpha_\beta$	The covariant, contravariant and mixed curvature tensors
$\lvert b_{\alpha\beta} \rvert$	The determinant of the second fundamental form
H, K	The mean and Gaussian curvature of the surface
k_n, τ	Normal curvature and the twist
$e^{\alpha\beta}, \varepsilon^{\alpha\beta}$	The permutation symbols for the cartesian and the

	curvilinear coordinates
$\Gamma_{\alpha\beta\gamma}, \Gamma^{\alpha}_{\beta\gamma}$	The Christoffel symbols of the first and second kind
$R_{\alpha\beta\gamma\upsilon}, R^{\alpha}{}_{\beta\gamma\upsilon}$	The Riemann_Christoffel symbols
$(\ )_{,\alpha}$	Partial differentiation with respect to surface coordinates
$(\ )\|_{\alpha}$	Covariant differentiation with respect to the first fundamental form of a surface
$\dot{(\ )}$	Material time derivatives
$\mathbf{v}, \mathbf{\Omega}$	The velocity and angular velocity vectors
$v^{\alpha}, v_{\alpha}, v$	The contravariant, covariant and normal components of the velocity vector
Ω^{α}, Ω	The tangential and normal components of the angular velocity
$n^{\alpha\beta}, q^{\alpha}$	The membrane stress tensor and the transverse shear force
$m^{\alpha\beta}, m^{3\alpha}$	The bending, twisting and normal mement tensors
$Y_{\alpha\beta}, \dot{a}_{\alpha\beta}$	The rate of change of membrane strains
$G_{\alpha\beta}$	The membrane strain tensor
$\beta^{\alpha\beta}$	The contravariant bending tensor
$C_{\alpha\beta}$	Covariant bending strain tensor
$Q^{\alpha\beta}$	An equivalent membrane stress tensor
$c_{\alpha\beta}$	Inextensional bending tensor
W	Strain energy per unit area
Q	Reciprocal surface

CHAPTER ONE

GENERAL INTRODUCTION

1_1 Introduction

Shell structures are curved surface structures designed for the purpose of covering large areas or to withstand great external loads. They have the same advantages compared to plates as arches compared to beams.

The origin of the word *shell* came from the hard covering of animals, eggs, nuts, or seeds. The role of this covering is to protect what is inside from external threats including loads.

Shell structures are used in many aspects of life including aerospace structures, shipbuilding, architecture, machine manufacture and components of physiological systems such as arteries and the cornea of the eye.

1_2 The principle of dimensions in structure

A line is one dimensional object in three dimensional space. It is one dimensional in that only one quantity is required to fix the position of a particular point on the line, for example the arc length from some fixed point. A line may be straight, curved in two dimensions (for example a circle) or curved in three dimensions (for example a spiral). In fig.(1_2.1), different representations of the line are shown to emphasise the aspect of dimensions.

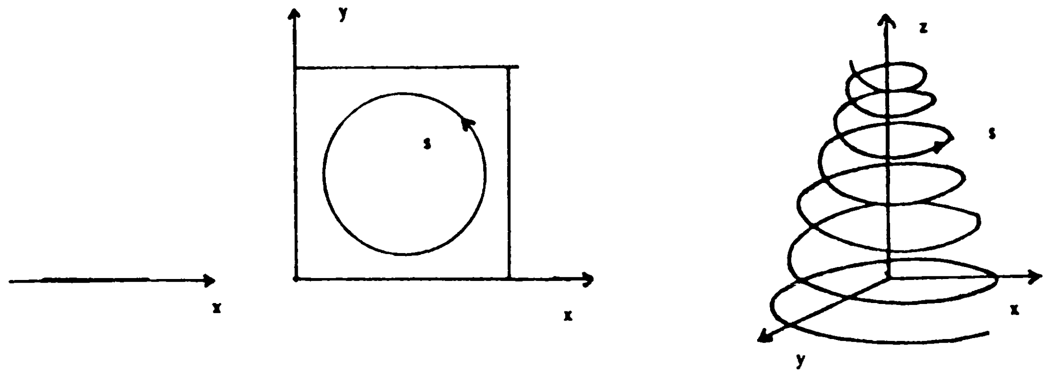


fig.(1_2.1) Different representations of a line

A line has zero cross_sectional dimensions and therefore it is not possible to physically construct a line in space. However, there are many examples of objects such as hairs and thin wires which have small cross_sectional dimensions and therefore approximate to lines in space.

Beams, columns and arches have cross_sectional dimensions which are perhaps 5 or 10% of their span. Engineers study their behaviour in terms of curvature and elongation of an imaginary line or axis which moves with the member as it deforms. Accordingly the change of curvature of the beam is assumed to be entirely determined by the effect of the bending moment, Calladine (1983).

change of curvature = Constant x *Bending moment.*

The above relation means that for a cantilever beam loaded at the remote edge by an external shear force the effect of transverse shear stresses is completely neglected in the deformation of the beam. This is known as Kirchhoff's hypothesis.

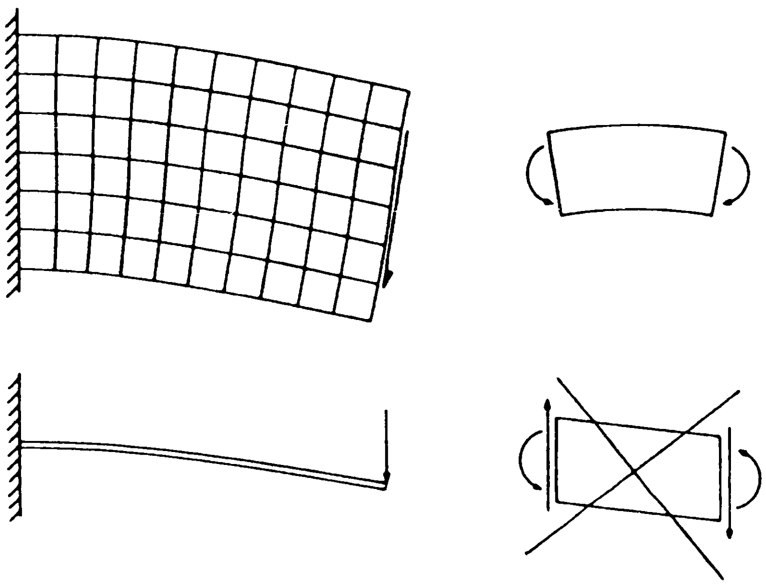


fig.(1_2.2) Beam idealisation "Calladine (1983)"

A surface is a two dimensional object in three dimensional space. It may be flat or curved. A surface has zero thickness and therefore it is not possible to construct a surface. However there are many structures such as egg shells, car bodies and concrete shells which have small thickness compared to their overall dimensions and radii of curvature and therefore approximate to a surface.

Flat plates and curved plates or shells have cross_sectional dimensions which are small compared to their overall dimensions and radii of curvature. According to the Kirchhoff and

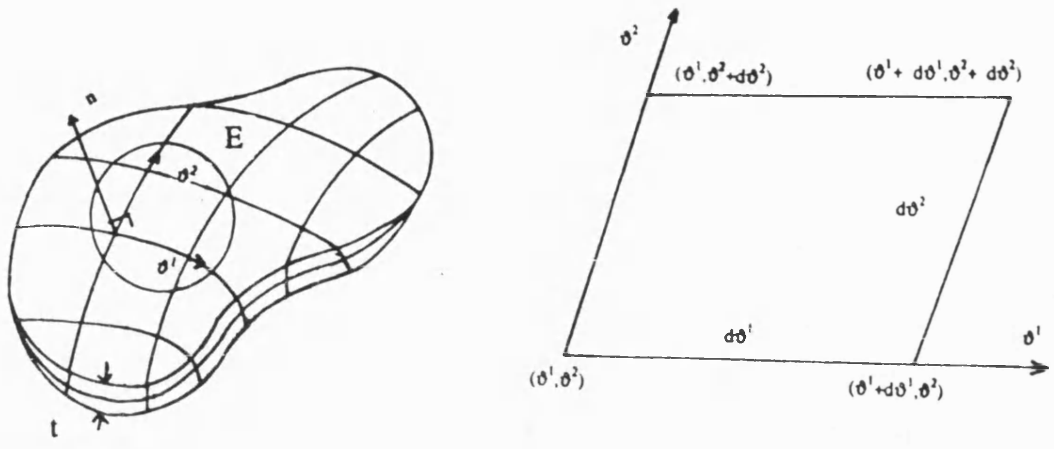
Kirchhoff_Love hypotheses for flat plates and shells respectively, deformation is referred to the middle surface which moves with the plate or the shell as they deform. Consequently transverse shear strain and normal shear strain will not affect the distortion of the middle surface. These hypotheses have constituted the corner_stone of shell theory and they will be presented in chapter 2.

1_3 Technical definition of shells

Shell structures are curved bodies that separate two different spaces, an inside space which represents the reason d'etre of the structure and the outside space. These two different spaces touch two different surfaces that constitute the shell's body.

The position of points which lie at equal distances from these two opposite surfaces constitute the middle surface of the shell. The distance between the opposite surfaces is known as the thickness of the shell.

In the theory of shell structures, including grid shells and ribbed shells, it is more appropriate to define a shell with respect to a reference surface rather than a middle surface and refer the axial forces, shear forces, twisting and bending moments to it. This will be explained and adopted later in this thesis.



A_ Portion of a shell *B_ The element E of a shell*

fig.(1_3.1) Geometrical representation of an element of shell

In the literature on shell theory there are two different classes of shells: thick shells and thin shells. Vlasov (1951) defines a shell as thin, if:

$$t/R_{min} \leq 1/30$$

where t, R_{min} are the thickness and the minimum radius of curvatures respectively. Shells for which this inequality is violated are called thick shells.

Most shell structures which are designed to be strong and adaptable to a broad range of applications have the property of being thin enough to be represented by surfaces and thick enough to resist compressive stresses.

Shells are in two forms according to their purposes: open surfaces such as cooling towers and domes with skylights and totally closed surfaces like some tanks and pressure vessels.

The difference in behaviour between open and closed shells gives rise to the question of rigidity, which in turn implies the necessity of providing open shells with boundary stiffeners under certain circumstances.

1_4 Mechanism of carrying external loads

External loads applied to shell structures are carried by means of a combination of stretching (*extension and compression*) and bending effects.

The stretching effect in a shell structure is studied by the use of the *membrane hypothesis.* Engineers usually consider this hypothesis as a starting point in the analysis, due to its effectiveness and sometimes accuracy of results. In this hypothesis the totality of load is assumed to be carried by in_plane stresses action only, in which stress couples and normal shearing forces are neglected.

Membrane theory is a particular mechanism for carrying loads, and can physically be modelled as a structural truss in which the joints of the members are made frictionless, Calladine (1983).

As far as an infinitesimally small element of a shell structure is concerned, membrane theory always leads to a statically determinate problem. Indeterminacy arises only when the shell element is subject to some specific boundary conditions. If the solution of the equilibrium equations satisfies the boundary conditions, the membrane theory can be considered a complete approximate solution of the problem at hand. However, if the equilibrium equations are insufficient for the discussion of the boundary conditions, we must seek another solution taking into consideration stress couples and normal shearing forces. This can be achieved by considering membrane theory as a particular solution of the problem and it may be supplemented by a homogeneous solution involving stress couples and normal shearing forces. Thus, it seems that the resulting problem can be called a mixed state of stress in which the bending stresses are assumed to be quantities of the same importance as the stresses in the tangential directions of the shell.

In most cases, it is more difficult to stretch a shell structure than bend it. This is because such structures have high mid-surface rigidity compared to their bending rigidity. Consequently, the general character of deformation is that lengths and angles on the surface change little during deformation.

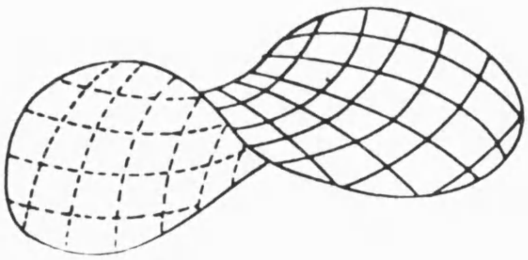
In the inextensional deformation theory of shells it is assumed that there is zero change in lengths and angles on the surface. A shell will try and deform in an inextensional mode and therefore a designer should always try and ensure that inextensional modes are suppressed by the correct shell geometry and proper boundary support.

The importance of the form of the structure and hence its geometry in producing a rigid structure may be better explained by the following example.

Consider a hyperbolic paraboloid shell, circular in plan and supported by a vertical support all around its perimeter. Its reference surface is given in Cartesian coordinates by

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2} \qquad (1\_1.1)$$

and we shall show that if $a = b$, then it is possible for the shell to undergo inextensional deformation.



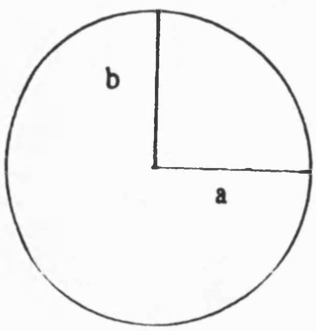


fig.(1_4.1) *A hyperbolic paraboloid shell*

A point P on the undeformed surface has the plane coordinates x, y and z. The line element on the surface takes the form

$$\delta s^2 = \delta x^2 + \delta y^2 + \delta z^2.$$

The coordinates of the point P on the deformed surface become (x+u), (y+v) and (z+w), and we write the line element of the surface as

$$\delta s'^2 = (\delta x + \delta u)^2 + (\delta y + \delta v)^2 + (\delta z + \delta w)^2$$

where u, v and w are the components in x, y and z directions of a small displacement vector v. The change in the line element, when neglecting higher orders for small displacement, is

$$= 2\ (\delta x\ \delta u + \delta y\ \delta v + \delta z\ \delta w).$$

For z, u, v and w functions of x and y, we write the change as

$$2\ \delta x \left(\frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y \right) + 2\ \delta y \left(\frac{\partial v}{\partial x} \delta x + \frac{\partial v}{\partial y} \delta y \right)$$

$$+ 2 \left(\frac{\partial z}{\partial x} \delta x + \frac{\partial z}{\partial y} \delta y \right) \left(\frac{\partial w}{\partial x} \delta x + \frac{\partial w}{\partial y} \delta y \right) \qquad (1\_1.2)$$

for any ratio ($\delta x : \delta y$), if the structure is deformed inextensionally. Equation (1_1.2) can be written in the form

$$\left(\frac{\partial u}{\partial x} + \frac{\partial z}{\partial x} \frac{\partial w}{\partial x} \right) \delta x^2 + \left(\frac{\partial v}{\partial y} + \frac{\partial z}{\partial y} \frac{\partial w}{\partial y} \right) \delta y^2$$

$$(1\_1.3)$$

$$+ \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial z}{\partial y} \frac{\partial w}{\partial x} + \frac{\partial z}{\partial x} \frac{\partial w}{\partial y} \right) \delta x\ \delta y = 0$$

which is equivalent to the following three equations

$$\left(\frac{\partial u}{\partial x}+\frac{\partial z}{\partial x}\,\frac{\partial w}{\partial x}\right)=0$$

$$\left(\frac{\partial v}{\partial y}+\frac{\partial z}{\partial y}\,\frac{\partial w}{\partial y}\right)=0 \qquad (1\_1.4)$$

$$\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial z}{\partial y}\,\frac{\partial w}{\partial x}+\frac{\partial z}{\partial x}\,\frac{\partial w}{\partial y}\right)=0.$$

Differentiating, the first equation twice with respect to y, the second equation twice with respect to x and the third equation with respect to x and then y, adding the two first equations and subtracting the third, we finally get

$$z_{,yy}\,w_{,xx}-2\,z_{,xy}\,w_{,xy}+z_{,xx}\,w_{,yy}=0. \qquad (1\_1.5)$$

Forming the second derivatives of z from (1_1.1), we write

$$z_{,yy}=\frac{-2\,c}{b^2}\quad,\quad z_{,xx}=\frac{2\,c}{a^2}\quad,\quad z_{,xy}=0.$$

Substituting the above expressions in (1_1.5), we finally have

$$\frac{-2\,c}{b^2}\,w_{,xx}+\frac{2\,c}{a^2}\,w_{,yy}=0. \qquad (1\_1.6)$$

This is a second order partial differential equations in the normal displacement, it is of hyperbolic type. It has the following solution

$$w=a^2y^2+b^2x^2+\text{const.} \qquad (1\_1.7)$$

To compute the two other tangential components of displacement, we use the two first equations in (1_1.4), and write

$$\frac{\partial u}{\partial x} = - \frac{\partial z}{\partial x}\frac{\partial w}{\partial x} \quad , \quad \frac{\partial v}{\partial y} = - \frac{\partial z}{\partial y}\frac{\partial w}{\partial y} \tag{1_1.8}$$

where

$$\frac{\partial z}{\partial x} = 2\, c\, \frac{x}{a^2} \quad , \quad \frac{\partial w}{\partial x} = 2\, b^2 x$$
$$\frac{\partial z}{\partial y} = -\, 2\, c\, \frac{y}{b^2} \quad , \quad \frac{\partial w}{\partial y} = 2\, a^2 y. \tag{1_1.9}$$

Substituting the corresponding values from (1_1.9) into (1_1.8) and performing simple integrations, we write for $a = b$

$$u = - \frac{4}{3} c\, x^3 + c_1$$
$$v = \frac{4}{3} c\, y^3 + c_2 \tag{1_1.10}$$

where c_1 and c_2 are constants of integration.

Equation (1_1.10) shows the existence of a mechanism in the structure, the supports move in the plan directions due to u and v. Thus, this allows the movement of the shell at the top of the structure up and down. Investigation of both equations, shows the dependence of the inextensional displacements u and v on the rise c of the structure which represents the concavity and convexity and hence the form of the shell.

1_5 The structure of the thesis

Having briefly discussed the way in which shells carry loads and the effect of geometry and boundary conditions on designing a rigid shell, the reminder of this thesis is organised as follows:

In chapter two, a brief history concerning shell theory is presented to emphasise the different approaches adopted in the analysis of such structures. For a better understanding of the notation used throughout the present work, the author found it important to include chapter three to explain tensor notation and some aspects in geometry.

Chapter four contains the statics of shells in two dimensions using the principle of a reference surface after the manner of Love.

In chapter five, the principle of deformation is investigated using a new parameter called the angular velocity. The deformed element of surface is fully defined by this new parameter, that is whenever the element of the reference surface undergoes a motion the three components of angular velocity define completely this motion.

Chapters six and seven contain respectively the constitutive equations and solution of equations and boundary conditions. Chapter six has been written just for the sake of completing the theory and to permit a brief discussion of the solution of equations and boundary conditions that follow in chapter seven.

In chapter eight the membrane hypothesis and the inextensional deformation are obtained systematically from the general theory presented in chapter four and five respectively. Also their relationships are discussed in a detailed manner.

In chapter nine the problem of rigidity of closed shells is discussed using the Cohn-Vossen theorem from differential geometry and its possible extension to cover wide variety of surfaces.

Chapter ten is concerned with the rigidity of open shells where the concept of inextensional deformation is applied to three different surfaces and the consequent boundary support are adopted.

Finally chapter eleven is reserved for the general discussion of results.

CHAPTER TWO

HISTORY OF SHELL THEORY

2_1 Introduction

In the author's opinion, there are two different ways to write about or describe the history of shell theory. The first can be very long if one goes into details and describes every contribution however small it may be. Alternatively, writing a short history may be fruitful if one can successfully describe in a clear manner the important points of development of the theory. In this chapter, it is intended to adopt the second approach and to divide the developments that shell theory has gone through into two periods of time.

2_2 History of the theory of shell up to 1940

Bouma (1962) states that one of the earliest study of the theory of shell structure was an investigation of the membrane stresses of shells of revolution published by Lamé and Clapeyron in (1828).

However, as the understanding of arches was the result of well established beam theory, shell structures were also preceded by the theory of plates. Consequently a short description of the development of the theory of plates is necessary.

In the literature on plates, two fundamental methods are used to solve the problems. According to Love (1944), Poisson & Cauchy around 1828 proposed a method to solve the problems of plates based on the expansion of all quantities (displacements and stresses) in a power series of the distance from the middle surface of the plate. On the other hand, based on physical ideas, Kirchhoff in 1850 introduced few assumptions concerning the surface and its deformation and expressed the potential energy of the bent plate in terms of the curvatures produced in its middle surface. In addition, the equations of motion and boundary conditions were obtained from the principle of virtual work.

The assumptions according to Novozhilov (1959) are;

a)_The straight fibers of a plate which are perpendicular to the middle surface before deformation remain so after deformation and do not change their length.

b)_The normal stresses acting on a plane parallel to the middle surface may be neglected in comparison with the other stresses.

In a comparison between the two methods, Novozhilov (1959), shows that Kirchhoff method excels over the method of Poisson & Cauchy by its great clarity and physical meaning. It solves also the problem of formulation of the boundary conditions which constitutes one of the major problems found in the power series method. However, Kirchhoff's method is based on assumptions and therefore it is approximate and more accurate results cannot be obtained. Donnell (1976) stated that" ..the lines after deformation are in general no longer exactly normal to the middle surface because of transverse shear strains, no longer straight because these strains vary with the distance from the middle surface, and no longer have the same length because of transverse normal strains". In contrast, the power series method leads to an exact solution as soon as the series converge.

According to Love (1944), the approximate character of the expression of potential energy was reconsidered again by Gehring and Kirchhoff around 1860. The resulting potential energy per unit area consists of two terms: One quadrature function of the quantity defining the extension of the middle surface with a coefficient proportional to the thickness. The other quadrature function defining the flexure of the middle surface with a coefficient proportional to the cube of the thickness.

A direct application of the Gehring & Kirchhoff method was carried out by Clebsch to solve some particular plate problems. He formed the equations of equilibrium of the plate in terms of stress resultants and stress couples. These equations are of two characters, one set involving tensions and in_plane shearing stresses and the second set involving stress couples and vertical shearing forces.

Based on the previous work, and particularly on Clebsch's method, a derivation of the general bending theory of shells was attempted for the first time starting from the general equations of elasticity by H.Aron in 1874. He expressed the geometry of a surface using the two parameters idea after the manner of Gauss.

He obtained an expression for the potential energy similar to that which Kirchhoff obtained for plates. According to Novozhilov (1959), however, Aron's development was not strictly correct, and inaccuracies were corrected by Love in 1888. In this latter work, Love proposed a theory for shells based on the same assumptions of Kirchhoff for the plates, and from which the name Kirchhoff_Love assumptions follows.

Pavlovic (1978) reported the achievements of E.Mathieu in his memoir in 1881 based on the work of Poisson. By studying the vibration of bells, E.Mathieu concluded that it is not possible to choose the meridian and the thickness of the shell in a way to end up with a vibration consisting only of the normal component of displacements. The reason is that the equations describing the normal and tangential displacements are of non_independent character due to the curvatures and the magnitudes of these two different displacements are usually of the same order.

The controversy concerning the behaviour of thin hemispherical bowls between Lord Rayleigh in 1881 and Love's famous work in 1888, was not solved until 1890 by Bassat & Lamb. In his investigation Lord Rayleigh, concluded by physical reasoning that the middle surface of the bowl remains unstretched and pure bending would dominate the character of vibration of the thin bowl. In contrast, Love obtained expressions, as described before, for the potential energy composed from two different terms. This led Love to conclude that stretching would be the dominant character of deformation in the vibration as long as the shell is very thin, and that the Rayleigh method does not satisfy the free edge of the shell.

By combining the physical understanding and the rigor of mathematics, Bassat & Lamb combined their efforts to explain the impact of the two different aspects of the modes of vibration. They concluded that a vibrating shell could have a very high extensional strains, but confined only to a narrow zone from the free edge. Far from that, towards the mid_length of the shell, it would behave inextensionally as Lord Rayleigh suggested. In this manner the magnitude of extensional strains would satisfy the boundary condition of the free edge, and the narrow zone of its influence explains the dominant character of inextensional mode of vibration in the remaining part of the shell.

Although the contribution of Bassat & Lamb explained some misunderstanding of the physical behaviour of shells, Love's theory still contains some inconsistencies. According to Pavlovic (1978), in Love's first approximation some terms containing t^3 are retained and some others are neglected.

Bouma (1962) reported that, until 1940 the development was concerned only with simple shells of revolutions, such as the spherical shell, the cylindrical shell and the hyperbolic paraboloid shell.

2_3 Development since 1940

As mentioned before, Love's first approximation is not free from inadequacies and many refinements have been suggested to correct its contents. Among these refinements, (Kraus (1967) and Pavlovic (1978)), the following are suggested:

a)_ All terms involving t up to $t^3$are retained in the equations

b)_ Some modifications in the early assumptions have been brought in what is known as Love's second approximation. These are first, the effect of transverse normal stresses is not to be neglected, and then the effect of transverse shear stresses is also to be included.

An analysis made by Koiter (1959) shows that the above refinements have brought little to Love's first approximation, in the sense that they are based on an approximate assumptions. These refinements are in most cases of the same order of magnitude as the errors which remain on account of the basic assumptions. An exception to the previous refinements is the work of Sanders (1959), who removed the inconsistencies of Love's first approximation with regard to rigid body motion strains. The resulting set of shell equations is known in the modern literature as Sanders_Koiter equations. According to Budiansky & Sanders (1963) this set of equations constitute a first order theory, in that the deformed state of the shell is determined entirely by the deformed configuration of its middle surface.

The resulting set of shell's equations is of order eight with the well known set of four boundary conditions to be satisfied at the edge(s) of the shell. Despite the approximations, this set of equations turns out to be too formidable that an analytic solution can rarely be obtained for it.

Along side to the classical Love theory of shells, a three_dimensional exact method was attempted by several authors. This is based on the fact that theoretically speaking, the states of stress and strain in shell structure are three_dimensional no matter how thin the shell may be. Following this line, the works of Zerna (1962), Naghdi (1963) and Green & Zerna (1968) are to be distinguished. Zerna (1962) used the three_dimensional theory and obtained an exact linear theory based on only two fundamental assumptions. It should be noted, however, that the author has pointed out that his theory may suffer from defects with respect to the physical behaviour of shells. In Naghdi (1963) and Green & Zerna (1968) the authors derived the field equations on the basis of the three_dimensional theory but employed different approaches for the constitutive equations. Green and Zerna (1968) used the constitutive equations of the linear theory from the three_dimensional equations of classical elasticity and applied them to some particular shell geometries where simplifications are available. Naghdi (1963) derived the constitutive equations on the basis of a variational theorem and gave an extensive analysis and discussion of the existing simplifications and approximations in the general linear theory of shells.

A mathematically exact analysis of states of stress and strain in a shell through the three_dimensional theory leads to insurmountable difficulties. Most of these difficulties sprout from the physical behaviour characterized in the constitutive equations of the shell. Therefore, the idea of two_dimensional theory for thin shell presents itself in a natural way. It seems that the main target of every shell theory, is to develop a two_dimensional theory, which will be appropriate in view of the thinness of the shell, and allows solutions of particular problems. In order to achieve this transition, assumptions and approximations must be introduced in the analysis.

2_4 Two-dimensional shell theory

There are several methods through which a development of two_dimensional theory can be achieved. Three of these methods are described in sections 2_4.1, 2_4.2 and 2_4.3.

2_4.1 Asymptotic expansions

This method of approach begins with the three_dimensional equations of elasticity, and then uses asymptotic expansions. This means expanding the solution of three_dimensional equations of elasticity theory with respect to some small parameter related to the thickness of the shell. The shell structure in this procedure is separated into two classes of problems, which are respectively the interior class problems and the boundary layer class problems.

Johnson & Reissner (1958) and Reiss (1960) both considered the particular case of rotationally symmetric deformation of circular cylindrical shell. Reissner (1960) also applied the expansion procedure to the case of shell of revolution under symmetrical deformation. Reiss (1962) extended his previous work so as to include also unsymmetrical deformations. According to Green $(1962)_1$, in the case of symmetrical deformations of cylindrical shells an expansion procedure in terms of one characteristic parameter is probably satisfactory, but for unsymmetrical deformations more than one type of expansion is possible.

Based then on the previous work, Green $(1962)_{1,2}$ attempted an approximation for the general case of shell theory not restricted to a special system of coordinates. The first part of the work was concerned with the interior problems of the shell where four distinct cases are obtained. The second part of the work was concerned with the boundary layers of the shell. He reported that ".. the equations of the interior problems are not in general uniformly valid on to any boundary surface S of the shell, which is normal to its middle surface, in the sense that it is not possible to satisfy arbitrarily specified boundary conditions on S.".

Also motivated by the previous works, a stream of papers concerned with the foundation of shell theory as two_dimensional differential equations and boundary conditions for the determination of three_dimensional states of stress and displacements in elastic bodies has followed. Among these we mention the works of Gol'denveizer (1962,9), Reissner (1963,4,9), Green & Naghdi (1965) and Green & Laws (1966).

In a survey of recent progress on the foundation and basic equations of shell theory, Koiter (1969) argued that difficulties, apparently to assess the results, arise from the availability of variety of asymptotic expansions and also from the different type of asymptotic expansions that have to be used in different problems (depending on geometry, boundary conditions and surface loads). Consequently the resulting theory is formulated by a number of different sets of equations, where one has to select the appropriate set for the problem at hand. According to Koiter (1969), the above reasoning corresponds to the various possible simplifications of Love's equations. Therefore the asymptotic approximations approach based on the three_dimensional theory of elasticity, provides in a certain sense a justification of the classical shell theory.

2_4.2 Variational derivation

The variational derivation, also called the energy method, is another way by which the equations of the three_dimensional states of stress is reduced to a two_dimensional systems of stress and displacement equations. In this method of approach, the classical principles of variation in elasticity are used. These are the principle of minimum potential energy or called also minimum principle for displacements and the principle of minimum complementary energy or called minimum principle for stresses.

Along this line, we mention first the work of Koiter (1959) by which a verification of Love's theory as a first approximation for thin shell theory is obtained. Also Koiter (1961), using variational method in obtaining the equilibrium equations and boundary conditions, set a systematic simplification of the equations of linear shell theory. By observing the order of magnitude of the ratio of the maximum flexural strain to the maximum extensional strain in the equilibrium and compatibility equations, He obtained nine cases of states of stress. Also important is the work of Reissner (1962,70) where the possibility of contracting the number of boundary conditions in the case of adding a vanishing mid_surface strains to the condition of no transverse shear strains through the thickness of the shell.

2_4.3 The Cosserat surface

In parallel with the previous attempts to derive a two dimensional shell theory, a direct two_dimensional model for the shell structures has been proposed for the first time by Duhem in 1893 and developed later by the brothers E. and F. Cosserat in 1909. The basic principles of this two_dimensional model is to consider an embedded surface in an Euclidean 3_space, to every point of which a vector (or several vectors) called the deformable director(s) is attached. These deformable directors are considered invariant in length under rigid body motion. (A rigid body motion is a motion that conserves lengths and angels on the surface during the motion). This method of approach which is a mathematical modelling of a physical problem, and called after the brothers E. and F. Cosserat remained unknown until fairly recently.

According to Ericksen & Truesdell (1958), Duhem argued " A body is to be regarded as a collection not only of points but also of directions associated with the points, these vectors which we shall call the directors of the body, are susceptible of rotations and stretches independent of the deformation of material elements...". The Cosserat brothers also remarked that ".. In one and two dimensions, the model serves admirably to represent the twisting of rods and shells in addition to their bending.". In this paper Ericksen & Truesdell (1958) developed an exact general theory for strain measures for rods and shells, and also obtained an exact stress equilibrium equations. Two years later these strain measures was also reproduced in the monograph of Truesdell & Toupin (1960).

Several attempts came later to develop the theory of oriented bodies. However, it was not until the paper by Green, Naghdi & Winwright (1965), that a full and general theory was set and developed, comprising both linear and nonlinear constitutive equations for the elastic Cosserat surfaces.

Following this development, several attempts came later for specialization of the theory. Among these, we mention the work of Balaban, Green & Naghdi (1967) concerning a general nonlinear theory for simple force multipoles. In this theory the notion of directors is completely omitted and the basic kinematic variables of the theory are **r** (the position vector of the surface) and its first and second derivatives. Also interesting is the work of Green & Naghdi (1968), Green & Naghdi (1969) considering several other aspects of the Cosserat surface, including the linear theory of an elastic Cosserat plate.

Finally, there is a special theory called the restricted theory derived by Naghdi (1972) which is a special case of the nonlinear theory of Cosserat surface with omitted directors. This restricted theory bears on the classical theory of shells. According to Green & Naghdi (1967) the theory of Cosserat surface may be regarded as an exact theory of deformable surfaces which is also applicable to shells.

2_5 Criteria for simplification of shell's equations

In the literature of shell structures, three main factors are usually found to be crucial for simplification of the usual eighth order sets of differential equations. These are respectively, the geometry of the structure, the applied external loads and the boundary conditions.

2_5.1 The effect of geometry

The name shell structure has been given to some particular surfaces having a particular characteristic. This particularity is represented by the geometry of the surface. Plates are flat surfaces, once curved they become shells of a certain particular type. The effect of endowing such curvature to the plates is noticed in their greater bearing load capacity. The choice of a particular type of curvature goes with the necessity to withstand a particular type of external load. In this respect the scalar known as the Gaussian curvature of the surface plays an important role in the structure, its sign governs the behaviour of the structure.

Calladine (1983) has stated that " In the theory of shells it seems that the governing equations are never rendred hyperbolic by the material properties alone, but they can be hyperbolic in some cases as a consequence of *geometrical* properties of the shell surface. In this sense the problem of shell structures may be said to be dominated by the geometry of the surface of the shell.". The effect of geometry in the simplification does not need to go as far as changing the sign of Gaussian curvature. However, as remarked by Lamb (see Pavlovic (1978)), the dominance of the stretching or bending and hence different states of stresses can be best explained by varying the curvature of the shell from almost zero (very shallow spherical cap) to nearly 4π (closed sphere). Vlasov (1951), described a very practical case concerning the reduction of the eighth order equations of shells to two simultaneous equations of the fourth_order in terms of the stress and displacement functions, when the shell is shallow.

2_5.2 The effect of external load

In many practical problems the external loads applied to the shell have the same symmetry as the shell itself. This helps in the sense that the stresses will be independent of one of the coordinates, and all derivatives with respect to that coordinate will disappear. Consequently, a simpler set of equations can be obtained. However, if there is an abruptly changing character of the external loads on the shell or of the shell itself, this leads to the variation of the states of stress in the shell, where different solutions have to be considered for the same structure.

2_5.3 The effect of boundary conditions

The specification of boundary conditions defines in some cases the state of stress on the edge of the shell and in other cases the displacement at the edge of the shell. The rigid clamping of the edges of the structure induces bending stresses at least over a narrow zone near the boundaries. Also preventing the structure from undergoing inextensional deformations by prescribing suitable boundary conditions could effect the possibility of a membrane shell. The details of this key factor in the analysis is discussed in the chapters concerned with the membrane theory and inextensional deformation.

2_6 States of stress

To sum up the effect of the three factors discussed in section 2_5.1, 2_5.2 and 2_5.3, Koiter (1961) established two important parameters by which a simplification of the equations of shell theory is achieved. This simplification leads always to some particular states of stress on the shell such as membrane theory, inextensional bending, shallow shell theory and generalized plane stresses..etc. He established the equations of compatibility and equilibrium and called them A, B and C, D in the normal and tangential directions respectively. Examination of these equations permits the estimation of the order of magnitude of the various extensional and bending terms. The state of stress in a certain region of shell can be characterized by means of two parameters. The first parameter is the order of magnitude of the ratio of the maximum (absolute) flexural strain to the maximum (absolute) extensional strain denoted by $\frac{\rho h}{\gamma}$. Comparing this parameter to unity, three possibilities may be distinguished

$$\frac{\rho h}{\gamma} \ll 1 \quad , \quad \frac{\rho h}{\gamma} = \emptyset\,(1) \quad \text{or} \quad \frac{\rho h}{\gamma} \gg 1. \qquad (2\_6.1)$$

In the above inequality, $\emptyset$ (1) means of the order of one.

The second parameter is the ratio of the square of the wave_length L of the pattern of deformation to the product of the minimum radius R and the thickness h. Comparing this parameter to the unity, again three cases are possible

$$\frac{L^2}{h\,R} \ll 1 , \quad \frac{L^2}{h\,R} = \emptyset\,(1) \quad \text{or} \quad \frac{L^2}{h\,R} \gg 1. \tag{2\_6.2}$$

This parameter is obtained by introducing the concept of wave length L of the deformation pattern on the middle surface through

$$|Y_{\alpha\beta}|_{\lambda}| = \emptyset\,(\frac{\gamma}{L}) \quad , \quad |\rho_{\alpha\beta}|_{\lambda}| = \emptyset\,(\frac{\rho}{L}) \tag{2\_6.3}$$

into the equations, A, B, C, D. The quantities $Y_{\alpha\beta}$ and $\rho_{\alpha\beta}$ are the in-plane and bending strains respectively.

With three values for each of the two ratios, nine cases are obtained. Neglecting bending or membrane terms in each equation depend on the following inequalities

if, $\frac{\rho h}{\gamma}\,\frac{L^2}{h\,R} \ll 1$, bending terms are dropped from the equation

if, $\frac{\rho h}{\gamma}\,\frac{L^2}{h\,R} \gg 1$, membrane terms are dropped from the equation.

In the following result's table, the letters A, B, C and D with indices γ and ρ indicate that only the membrane or the bending terms are retained respectively. A letter without index in the table indicates that both terms have been retained in the equations.

	$\frac{L^2}{hR} \ll 1$	$\frac{L^2}{hR} = O(1)$	$\frac{L^2}{hR} \gg 1$
$\frac{\rho h}{\gamma} \ll 1$	Eqs. A_γ, B, C, D_γ Equations of generalized plane stress	Eqs. $A_\gamma, B, C_\gamma, D_\gamma$ Exceptional case in which the equations of generalized plane stress and the equations of membrane theory of shells are satisfied simultaneously	Eqs. A , B, C_γ, D_γ Equations of the membrane theory
$\frac{\rho h}{\gamma} = O(1)$	Eqs. $A_\gamma, B_\rho, C_\rho, D_\gamma$ Superposition of the equations of generalized plane stress and the equations of bending of flat plates	Eqs. A , B_ρ, C, D_γ Equations of shallow shells	Eqs. $A_\rho, B_\rho, C_\gamma, D_\gamma$ Superposition of the equations of the membrane theory of shells and the equations of nearly inextensional bending of shells
$\frac{\rho h}{\gamma} \gg 1$	Eqs. A , B_ρ, C_ρ, D Equations of bending of flat plates	Eqs. $A_\rho, B_\rho, C_\rho, D$ Exceptional case in which the equations of bending of flat plates and the equations of nearly inextesional bending of shells are satisfied simultaneouly	Eqs. A_ρ, B_ρ C , D Equation of nearly inextensional bending

Table (1) : Summary of simplified shell equations after Koiter (1961).

2_7 Conclusion

It is noticed from the forgoing discussion, and with the help of the work of Koiter (1961), that major simplifications are obtainable in the analysis and design of shells, if one starts first by considering the geometry of the structure and the external applied loads. Usually the resulting equations will be of a simple character, and in many cases a single state of stress is found over a considerable area of the shell.

CHAPTER THREE

DIFFERENTIAL GEOMETRY

3_1 Introduction

A shell structure is defined geometrically by, the reference surface, the thickness of the shell and its boundary. The most important of these is the reference surface which defines the shape of the shell. The structural behaviour of shells is determined largely by the geometry of the reference surface and therefore any study of shell structures must start with the geometry of surfaces.

Classical differential geometry is the study of curved lines and surfaces in three_ dimensional Euclidean space. Differential geometry can also be applied to higher dimensional space but that does not concern us in the present work.

In what follows, it is intended to start with a three_dimensional analysis and then perform the dimensions's reduction to two_dimensional application on the surface.

3_2 Tensor notation

Tensor notation is used throughout the present work, due to the fact that the analysis of shell structures is not only a structural analysis problem but also a geometrical problem. The tensor notation allows structural quantities and geometrical quantities such as curvature and twist to be treated in the same way. Some, physical quantities fall into various classes such as scalars and vectors. However, other quantities, of which the stress and strain distribution in a solid are perhaps the simplest examples, are more complicated than vectors and need to be expressed using tensor notation. A vector is associated with a single direction while a component of stress is associated with two directions.

Kil'chevskiy (1963) has reported that " ..Particularly important is the problem of constructing quantities independent of the choice of the coordinate system. These quantities are termed invariants of coordinate transformations. Tensor quantities are the base for the construction of invariants.
..... Most invariants have a definite geometric or physical meaning. Invariants are the basis for the general analytical formulations of laws of physics, especially those of mechanics. The applications of tensor analysis to the geometry of surfaces are numerous, since here tensor analysis allows us to find expressions of geometric theorems in a simple and yet general form.".

Thus tensor equation does not refer to any particular coordinate system, in fact, if it holds in one system of coordinate it holds in all, through the concept of coordinate transformations.

Quantities like temperature or electric potential are represented by scalars, these will not be affected during the process of coordinate transformations. Whereas quantities like forces and displacements are represented by vectors. Scalars and vectors constitute tensors of order zero and one respectively. Quantities like stresses and strains are represented by second order tensors.

Vectors and tensors of higher orders change according to certain rules of transformation when the coordinates are changed.

3_2.1 Coordinate transformations

The process of coordinate transformation is one of the most important operations required, when facing particular problems concerning the geometry of shell theory. The following work is based on the introductions to tensor analysis given by, Eisenhart (1947), Green & Zerna (1968), Niordson (1985) and Chung (1988).

If we define points in the three_dimensional space by a system of general curvilinear coordinate ϑ^i (where the upper Latin index i takes the values 1, 2 and 3), the transformation of this system of coordinates to a new system, say Cartesian coordinates x^i, should obey certain rules of transformation.

The following work will be applicable to any other system of coordinates as long as the one-to-one relations between the system of coordinates and the points of space hold.

If p^i are three independent single_valued continuously differentiable functions of ϑ^i, the system of equations

$$x^i = p^i(\vartheta^1, \vartheta^2, \vartheta^3) \qquad (3\_2.1)$$

defines the new system of Cartesian coordinates. The functions p^i are independent only if the determinant

$$\left| \frac{\partial p^i}{\partial \vartheta^j} \right| = | p^i_{,j} | \neq 0 \qquad (3\_2.2)$$

where for brevity we write

$$p^i_{,j} = \frac{\partial p^i}{\partial \vartheta^j}. \qquad (3\_2.3)$$

If we assume that the transformation is reversible, then by solving (3_2.1) for ϑ^i we obtain

$$\vartheta^i = q^i(x^1, x^2, x^3). \qquad (3\_2.4)$$

The functions q^i are also independent, single-valued and continuously differentiable with respect to x^i. Differentiating (3_2.1), the differentials dx^i are

$$dx^i = p^i_{,j} d\vartheta^j = \sum_{j=1}^{3} p^i_{,j} d\vartheta^j. \qquad (3\_2.5)$$

In equation (3_2.5), the process of repeating the index j is called Einstein's summation convention. Whenever an index is repeated twice in the same term, summation is implied over the range of that index . Such an index is called a dummy. In any equation the dummy index can be replaced by any similar index without altering the results of the equation.

Equation (3_2.5) indicates how one differential in a given system of coordinates can be transformed to another differential in another system of coordinates. These differentials constitute a components of a contravariant type of tensors of order one. The contravariant components H^j are identified by an upper index and we write the linear transformation from the general curvilinear coordinates ϑ^i to the Cartesian coordinates x^i as

$$H^{*i} = p^i_{,j} H^j. \qquad (3\_2.6)$$

3_2.2 Scalar invariants. covariant vectors

Weatherburn (1950) stated that " In its wider sense the term ''invariant'' denotes any object which is not changed by transformation of coordinates". If we suppose that F is a continuously differentiable scalar function in the general curvilinear coordinates ϑ^i, Then the same scalar is represented by another function F^* in the Cartesian coordinates x^i, and we write

$$F^*(x^i) = F (\vartheta^i). \qquad (3\_2.7)$$

From (3_2.4), (3_2.7) becomes

$$F^*(x^i) = F (q^i(x^1_, x^2_, x^3)). \qquad (3\_2.8)$$

Therefore, differentiating with respect to x^i we get

$$F^*_{,i} = F_{,j} q^j_{,i} \qquad (3\_2.9)$$

where

$$q^j_{,i} = \frac{\partial q^j}{\partial x^i}. \qquad (3\_2.10)$$

Equation (3_2.9) shows how the gradient $F,_j$ is transformed when the coordinate system is changed from ϑ^i to x^i. This type of coordinate transformation is called a covariant transformation.
Therefore the differential $d\vartheta_j$ is transformed to dx_i as follows

$$dx_i = \frac{\partial q^j}{\partial x^i} \, d\vartheta_j. \qquad (3\_2.11)$$

However the differential dx_i and $d\vartheta_j$ are not equal to the change in coordinates which are equal to dx^i and $d\vartheta^j$. This explains why coordinates are given in superscripts.

We write in general the covariant transformation from the general curvilinear coordinates ϑ^i to the Cartesian coordinates x^i, as

$$H^*_i = q^j_{,i} \, H_j \qquad (3\_2.12)$$

where the asterisk indicates Cartesian coordinates.
Thus, equations (3_2.12) shows that the gradient of any scalar function is a vector of covariant type. A covariant vector is identified by a lower index. However, it is noted that not all quantities with an upper or lower indices are contravariant or covariant components of vectors. Thus for instance dx_i and dx^i are components of covariant and contravariant vectors, while x^i which represent the space coordinates are not.

The principle of covariant and contravariant vector loses its meaning, if we limit ourselves to the transformation of coordinates between the right_ handed orthogonal Cartesian coordinates, and we write $H^i = H_i$. Then in this context the word, 'type' which distinguishes between vectors and also tensors being covariant or contravariant will not exist any longer.

3_2.3 Tensors of the second order

Now, to generalize the concept of coordinate transformations from vectors (first order tensor) to tensors of second order, we write the transformation of the contravariant type of tensor of second order as

$$H^{*ij} = p^i_{,k}\, p^j_{,l}\, H^{kl}. \qquad (3\_2.13)$$

Analogously, a covariant type of tensor of second order is

$$H^*_{ij} = q^k_{,i}\, q^l_{,j}\, H_{kl}. \qquad (3\_2.14)$$

The third type of a second order tensor is the mixed tensor and is defined as

$$H^{*i}_{.j} = p^i_{,k}\, q^l_{,j}\, H^k_{.l}. \qquad (3\_2.15)$$

We add another type of second order mixed tensor called the Kronecker delta δ^i_j, which is defined in the Cartesian coordinates by the following formulae

$$\delta^i_j = \begin{cases} 1 & \text{when } i=j \\ 0 & \text{when } i\neq j. \end{cases} \qquad (3\_2.16)$$

If we apply (3_2.15) to the kroneker delta we obtain

$$\delta^{*i}_j = p^i_{,k}\, q^l_{,j}\, \delta^k_l = p^i_{,k}\, q^k_{,j} = \delta^i_j. \qquad (3\_2.17)$$

Consider the two equations (3_2.1) and (3_2.4)

$$x^i = p^i(\vartheta^1, \vartheta^2, \vartheta^3) \qquad , \qquad \vartheta^i = q^i(x^1, x^2, x^3)$$

If we substitute the second equation into the first one we write

$$x^i = p^i(q^i(x^1, x^2, x^3)) \qquad (3\_2.18)$$

Differentiation with respect to x^i, yields

$$x^i_{,j} = p^i_{,k}\, q^k_{,j}. \qquad (3\_2.19)$$

Comparison of (3_2.17) and (3_2.19) shows that the Kronecker delta δ^i_j is a mixed tensor of second order. In tensor notation, δ^i_j plays the role of substitutional operator and also has the characteristic of being constant in all coordinates systems,

$$\delta^{*i}_j = \delta^i_j.$$

We write also in the Cartesian coordinates

$$\delta_{ij} = \delta^{ij} = \delta^i_j = \delta^j_i = 0 \qquad (i \neq j)$$

$$\delta_{ij} = \delta^{ij} = \delta^i_j = \delta^j_i = 1 \quad (i=j,\ j \text{ not summed}). \qquad (3\_2.20)$$

Equations (3_2.6) and (3_2.12) are the basic rules of transformation of contravariant and covariant vectors from one system of coordinate to another. Equations (3_2.13), (3_2.14) and (2_2.15) represent the extension of those basic rules of transformation from vectors to tensors of second order.

The same basic rules can be extended to cover tensors of arbitrary order in the following manner

$$H^{*rsp..}_{.ijm..} = p^{r}_{,u} p^{s}_{,v} \; p^{p}_{,k} \ldots q^{k}_{,i} \; q^{l}_{,j} q^{t}_{,m} \ldots H^{uvk..}_{.klt..} \qquad (3\_2.21)$$

where according to (3_2.1) and (3_2.4), the functions p^i and q^i represent the matrix of transformation of coordinates and the inverse transformation. Thus , a tensor is called of order n, where n = u+v and u and v represent the covariant and contravariant indices, only if its components transform according to the rule of transformation given in (3_2.21), from one system of coordinates to another. From (3_2.21), it is noticed that if the components of a tensor vanish in one coordinate system they vanish in all.

3_2.4 Algebraic operations of tensors

First, let us start with the basic operations of algebra on tensors. Among these operations, we note that the addition and subtraction of tensors, which apply only to tensors of the same order and type, lead to tensors of the same order and the same type. Thus, these operations are equivalent to those used in the algebra of real numbers, i.e

$$C^i = A^i + B^i$$

$$C^{ij} = A^{ij} - B^{ij}.$$

The sum and the difference of tensors at the same point and of the same dimensions lead to a tensor of that dimension.

However, the multiplication of tensors leads to a tensor of higher order, i.e.

$$C^{ij}_{..k} = A^{ij} B_k.$$

It is also called the outer product of two tensors. Thus, the outer product of two tensors at one and the same point is a tensor with an order equal to the sum of the orders of the two tensors.

To the above operations, we add another operation called tensor contraction. According to a theorem given in Niordson (1985), any contracted tensor is a tensor of two orders less (one contravariant and one covariant). If we have the tensor $T^{ik}_{..j}$ of order three and we substitute j by i, then we write $T^{ik}_{..i}$. From the tensor notation, we know that the repeated index is to be summed and hence, resulting in a contravariant tensor of order one. Again, multiplication of the contravariant vector U^i by the covariant vector V_j results in a mixed tensor of second order $U^i V_j$. Substitution of j by i results in a scalar $U^i V_i$ equivalent to the inner product of U^i and V_j.

An equivalent operation in tensor notation to the division in real numbers is the rigorous quotient theorem. This theorem according to Spain (1965) is useful to ascertain whether a set of functions form the components of a tensor. Therefore, instead of checking whether these functions obey the tensor transformation rules, it is easier to use the quotient theorem which states " A quantity which, on inner product by any covariant (alternatively contravariant) vector, always gives a tensor is itself a tensor.".

A^{ijk} form a contravariant tensor of order three

$$A^{ijk}B_{ij}^{\cdot\cdot p} = C^{pk}$$

provided that $B_{ij}^{\cdot\cdot p}$ is an arbitrary mixed tensor of order three and C^{pk} a contravariant tensor of order two.

3_2.5 Special tensors

The distance ds between two adjacent points P and Q in an orthogonal Cartesian coordinate system x^i, in which x^i and $x^i + dx^i$ are their respective coordinates, is given by

$$ds^2 = dx^i\, dx^i. \qquad (3\_2.22)$$

Using equation (3_2.5) the transformation ds^2 from the general curvilinear coordinates ϑ^i, where P and Q assumed respectively the coordinates ϑ^i and $\vartheta^i + d\vartheta^i$ is

$$ds^2 = dx^i\, dx^i = p^i_{,l}\, p^i_{,j} d\vartheta^l d\vartheta^j. \qquad (3\_2.23)$$

Setting

$$g_{lj} = p^i_{,l}\, p^i_{,j} \qquad (3\_2.24)$$

it is found that

$$ds^2 = g_{lj} d\vartheta^l d\vartheta^j. \qquad (3\_2.25)$$

In equation (3_2.25), g_{ij} comprises nine functions of

$(\vartheta^1,\vartheta^2,\vartheta^3)$. The quadratic differential form in the second member of (3_2.25) is called the Riemannian metric, and the space which is characterized by such a metric is the Riemannian space. The quantity g_{ij} can be easily evaluated when the relations p^i between the Cartesian and curvilinear coordinates are known. Equation (3_2.24) shows that g_{ij} is symmetric quantity, then

$$g_{ij} = g_{ji} \tag{3\_2.26}$$

which also shows only six out of nine functions are independent functions. Thus, comparison of (3_2.24) with equation (3_2.19) suggests that in cartesian coordinates $g_{ij} = \delta^i_j$.

In equation (3_2.25), we have ds^2 as an invariant and $d\vartheta^j$ is arbitrary, then $g_{ij}\ d\vartheta^i$ is a covariant vector. Also in $g_{ij}d\vartheta^i$, we know that $d\vartheta^i$ is arbitrary, then g_{ij} must be a covariant tensor of second order. The reason that the tensor g_{ij} is called the fundamental metric tensor of the space, is its importance in the identification of the space and its metric character as it can be seen in (3_2.25).

$$g = \det\ (g_{ij}) > 0.$$

For Cartesian coordinates, we make use of (3_2.24) and find

$$g = |g_{ij}| = \left[\det\ (p^i_{,j})\right]^2.$$

The inverse of g_{ij}, i.e. the contravariant second order tensor g^{ij} is also symmetric and given by

$$g^{ij} = \frac{D^{ij}}{g} \tag{3\_2.27}$$

where D^{ij} is the cofactor of g_{ij}. The determinant g is function of the coordinates $(\vartheta^1, \vartheta^2, \vartheta^3)$, therefore it is not a scalar. Collecting both determinants of g_{ij} and g^{ij}

$$g = |g_{ij}| = \left[\det\,(p^i_{,j})\right]^2$$

$$\frac{1}{g} = |g^{ij}| = \left[\det\,(q^i_{,j})\right]^2. \qquad (2\_2.28)$$

From equation (3_2.27) follows

$$g_{ij}\, g^{jk} = \delta^k_i. \qquad (2\_2.29)$$

Also for the mixed second order tensor, we write

$$g^k_i = \delta^k_i.$$

Then, the three fundamental tensors are the covariant, contravariant and the mixed second order tensors, they are all symmetric. By using the tensor transformation rules, we write in the general curvilinear coordinates the components of these tensors as follows

$$g_{lj} = p^i_{,l}\; p^i_{,j} = p^i_{,l}\; p^t_{,j}\delta_{it}$$

$$g^{lj} = q^l_{,i}\; q^j_{,i} = q^l_{,i}\; q^j_{,t}\delta^{it} \qquad (3\_2.30)$$

$$g^l_j = q^l_{,i}\; p^i_{,j} = q^l_{,t}\; p^i_{,j}\delta^t_i = \delta^l_j.$$

For orthogonal coordinates, we write the special results

$$g_{ij} = g^{ij} = 0 \qquad \text{for } (i \neq j)$$

$$g^{11} = \frac{1}{g_{11}} \quad , \quad g^{22} = \frac{1}{g_{22}}$$

$$g^{33} = \frac{1}{g_{33}} \qquad , \qquad g = g_{11} g_{22} g_{33} \qquad (3\_2.31)$$

Before closing this section let us consider a special skew_symmetric tensors known as the ε_system, which have components in the general coordinates ϑ^i denoted by ε_{ijk}, ε^{ijk}. In the Cartesian coordinates, these components are denoted by e_{ijk} and e^{ijk} and are (Green & Zerna (1968)) given by

$e_{ijk} = e^{ijk} = 0$ when any two of the indices are equal

$= + 1$ when i,j,k is an even permutation of 1,2,3

$= - 1$ when i,j,k is an odd permutation of 1,2,3.

Thus, in the general curvilinear coordinates we write

$$\varepsilon_{ijk} = p^r_{,i} \; p^q_{,j} \; p^s_{,k} \; e_{rqs}$$

$$\varepsilon^{ijk} = q^i_{,r} \; q^j_{,q} \; q^k_{,s} \; e^{rqs}. \qquad (3\_2.32)$$

Using (3_2.28), the relationship between the different components of the e system in the Cartesian coordinates to those of the general curvilinear coordinate is given by

$$\varepsilon_{ijk} = e_{ijk} \sqrt{g}$$

$$\varepsilon^{ijk} = e^{ijk} / \sqrt{g}. \qquad (3\_2.33)$$

The origin of the ε system, which is also called the permutation symbols in some other text books, Chung (1988), is the cross product of unit vectors.

3_2.6 Associated tensors

The process of raising and lowering indices of the components of vectors can be developed to cover also tensors of higher order. Consider first the contravariant vector A^i, then $g_{ij}A^i$ is a covariant vector, which is equivalent to A_j, then we write

$$A_j = g_{ij}A^i. \tag{3_2.34}$$

Then equation (3_2.34) gives the possibility of substitution and the operation is known as lowering of indices. The vector A_j will be called the associated vector to A^i and vice versa. Since the determinant g is different from zero, then from (3_2.29) we take

$$A^i = g^{ij}A_j. \tag{3_2.35}$$

The above operation is known as the raising of indices.

This way of producing associated vectors can be extended as we mentioned before to tensors of higher order. However, care must be taken when dealing with mixed tensors to ensure consistency.
If we have the covariant tensor of the second order A_{ij} then, multiplying it by g^{rs} and contracting with respect to the first index we take

$$A^{s}_{\cdot j} = g^{is}A_{ij}. \tag{3_2.36}$$

Clearly, it is the first index which has been raised, the dot before the second index is to emphasize that and it is unnecessary only if A_{ij} is symmetric. The process of raising the second index will be

$$A_{r}^{\cdot s} = g^{sj} A_{rj}. \qquad (3\_2.37)$$

Similarly the process of lowering the second order contravariant tensor A^{ij} is

$$\left.\begin{aligned} A^{r}_{\cdot s} &= g_{sj} A^{rj} \\ A_{r}^{\cdot s} &= g_{jr} A^{js} \end{aligned}\right\} \qquad (3\_2.38)$$

Similar raising and lowering of indices can be performed on higher order tensors.

3_2.7 The base vectors

Consider the space x^i in which the coordinates of two adjacent points C and P are given respectively by x^i and $x^i + dx^i$, see fig.(3_2.1). **R** is a differentiable function expressing the position vector of C at x^i. In this case we have

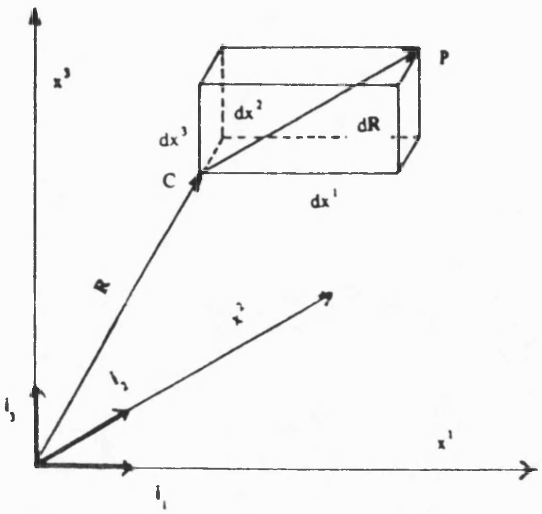


fig.(3_2.1) The position vector, Green & Zerna (1968)

$$\mathbf{R}(x^1,x^2,x^3)= x^i.\mathbf{i}_i. \qquad (3\_2.39)$$

Let dR be the vector $\overline{CP}$ then, using (3_2.39),

$$\mathbf{dR} = \frac{\partial \mathbf{R}}{\partial x^i} dx^i = \mathbf{i}_i \; dx^i \qquad (3\_2.40)$$

where $\mathbf{i}_i = \mathbf{i}^i$ are the constant unit vectors in the Cartesian coordinates x^i, their inner and outer products are given by

$$\mathbf{i}_r.\mathbf{i}_s = \mathbf{i}_r.\mathbf{i}^s = \mathbf{i}^r.\mathbf{i}_s = \mathbf{i}^r.\mathbf{i}^s = \delta^r_s$$

$$\mathbf{i}_r x\; \mathbf{i}_s = \mathbf{i}^r x\; \mathbf{i}^s = e_{rst}\mathbf{i}^t = e^{rst}\mathbf{i}_t. \qquad (3\_2.41)$$

If ds = |dR| is the length of the vector $\overline{CP}$, then from (3_2.40)

$$ds^2 = \mathbf{dR}\;.\mathbf{dR} = dx^i dx^i. \qquad (3\_2.42)$$

With the help of (3_2.1) The position vector transforms to the general curvilinear coordinates and becomes

$$\mathbf{R} = \mathbf{R}(\vartheta^1,\vartheta^2,\vartheta^3).$$

Thus, according to (3_2.5), using the chain rule

$$\mathbf{dR} = \frac{\partial \mathbf{R}}{\partial \vartheta^i} d\vartheta^i = \frac{\partial \mathbf{R}}{\partial x^i}\; \frac{\partial x^i}{\partial \vartheta^r} d\vartheta^r\;.$$

Thus, with the use of (3_2.40)

$$\mathbf{dR} = \mathbf{g}_r \; d\vartheta^r = \mathbf{g}^r d\vartheta_r \qquad (3\_2.43)$$

where

$$\mathbf{g}_r = \frac{\partial \mathbf{R}}{\partial x^i}\; \frac{\partial x^i}{\partial \vartheta^r} = \frac{\partial x^i}{\partial \vartheta^r}\mathbf{i}_i \;,\quad \mathbf{g}^r = \frac{\partial \vartheta^r}{\partial x^i}\mathbf{i}^i. \qquad (3\_2.44)$$

From (3_2.43), $\mathbf{g}_r$ can also be written as partial derivatives of the position vector **R**

$$\mathbf{g}_r = \mathbf{R},_r = \frac{\partial \mathbf{R}}{\partial \vartheta^r}. \qquad (3\_2.45)$$

$\mathbf{g}_r$ and $\mathbf{g}^r$ are called respectively the covariant and contravariant base vectors of the space ϑ^i. The base vectors are not unit vectors since they are multiplied by the factors $\frac{\partial x^i}{\partial \vartheta^r}$ and $\frac{\partial \vartheta^r}{\partial x^i}$ which represent the transformation matrices $p^i_{,r}$ and $q^r_{,i}$. They represent the rate of change of the position vector. Equation (3_2.43) shows that the base vectors $\mathbf{g}_r$ and $\mathbf{g}^r$ are obtained from the constant unit vectors **i**'s by transformations similar to those of the covariant and contravariant vectors in (3_2.6) and (3_2.12). Equation (3_2.42), now becomes

$$ds^2 = d\mathbf{R} . d\mathbf{R} = \mathbf{g}_r d\vartheta^r . \mathbf{g}_s d\vartheta^s. \qquad (3\_2.46)$$

Comparison between (3_2.44) and (3_2.25) shows that the covariant and contravariant base vectors are related with the tensors g_{ij}, g^{ij} and g^i_j by the linear relations

$$\begin{aligned} \mathbf{g}_r . \mathbf{g}_s &= g_{rs} \\ \mathbf{g}^r . \mathbf{g}^s &= g^{rs} \\ \mathbf{g}^r . \mathbf{g}_s &= g^r_s = \delta^r_s. \end{aligned} \qquad (3\_2.47)$$

Thus

$$\mathbf{g}_r = g_{rs} \mathbf{g}^s, \quad \mathbf{g}^r = g^{rs} \mathbf{g}_s \qquad (3\_2.48)$$

A useful result to be noted from (3_2.47), is that the contravariant base vectors $\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3$ are respectively perpendicular to the planes enclosed between $\mathbf{g}_2\mathbf{g}_3$, $\mathbf{g}_3\mathbf{g}_1$, $\mathbf{g}_1\mathbf{g}_2$. From equation

(3_2.46), the line element ds is

$$ds^2 = d\mathbf{R} \cdot d\mathbf{R} = g_{rs} d\vartheta^r d\vartheta^s. \tag{3_2.49}$$

Also the line elements along the coordinate curves are

$$ds_i = g_i d\vartheta^i \quad \text{(i is not summed).} \tag{3_2.50}$$

The magnitudes of the covariant and contravariant base vectors are given by

$$|g_i| = \sqrt{(g_i \cdot g_i)} = \sqrt{g_{ii}}$$

$$|g^i| = \sqrt{(g^i \cdot g^i)} = \sqrt{g^{ii}} \quad \text{(i not summed)} \tag{3_2.51}$$

and the magnitude of the line elements is

$$ds_i = \sqrt{(g_{ii})} d\vartheta^i \tag{3_2.52}$$

The vector products of the covariant and contravariant base vectors can be obtained using (3_2.41) and (3_2.44)

$$g_r \times g_s = \varepsilon_{rst} g^t$$

$$g^r \times g^s = \varepsilon^{rst} g_t. \tag{3_2.53}$$

The triple product is given by

$$\left[g_r g_s g_t\right] = \left[g_s g_t g_r\right] = \left[g_t g_r g_s\right] = \varepsilon_{rst}$$

$$\left[g^r g^s g^t\right] = \left[g^s g^t g^r\right] = \left[g^t g^r g^s\right] = \varepsilon^{rst}. \tag{3_2.54}$$

Using (3_2.33), in particular we write

$$\left[g_1 g_2 g_3\right] = \sqrt{g} \quad , \quad \left[g^1 g^2 g^3\right] = 1/\sqrt{g}. \qquad (3\_2.55)$$

The element of area dS_1 on the $\vartheta^1\_$ surface is

$$dS_1 = |ds_2 x\ ds_3| = |g_2 x\ g_3|\ d\vartheta^2 d\vartheta^3.$$

Then, upon using (3 _2.51) and (3_2.53) the element area becomes

$$dS_1 = \sqrt{(gg^{11})} d\vartheta^2 d\vartheta^3. \qquad (3\_2.56)$$

In general we write

$$dS_i = \sqrt{(gg^{ii})} d\vartheta^j d\vartheta^k \quad (i \text{ not summed, } i \neq j \neq k). \qquad (3\_2.57)$$

The volume element is given by

$$d\tau = ds_1.(ds_2 x\ ds_3) = [g_1 g_2 g_3] d\vartheta^1 d\vartheta^2 d\vartheta^3$$

$$= \sqrt{(g)} d\vartheta^1 d\vartheta^2 d\vartheta^3. \qquad (3\_2.58)$$

The inner product of the entities v^r, which represents a contravariant components of vector and transforms according to the contravariant rule of transformation, and g_r, which represents a covariant base vector and transforms according to the covariant rule of transformation, yields an invariant quantity represented by the vector v. Then using (3_2.47)

$$v = v^r\ g_r = v_r\ g^r \qquad (3\_2.59)$$

where

$$v_r = g_{rg} v^g \ , \quad v^r = g^{rg} v_g. \qquad (3\_2.60)$$

The scalar product of the vector v by a similar vector w is

$$\mathbf{v} \cdot \mathbf{w} = v^r \, w^s \mathbf{g}_r \cdot \mathbf{g}_s = v_r \, w_s \mathbf{g}^r \cdot \mathbf{g}^s. \tag{3_2.61}$$

Using equations (3_2.47), (3_2.61) becomes

$$\mathbf{v} \cdot \mathbf{w} = v^r w^s g_{rs} = v_r w_s g^{rs} = v_r w^r = v^s w_s. \tag{3_2.62}$$

The magnitude of the **v** vector is

$$|\mathbf{v}| = \sqrt{(\mathbf{v}.\mathbf{v})} = \sqrt{(v^r v^s g_{rs})} = \sqrt{(v_r v_s g^{rs})} = \sqrt{(v_r v^r)} \tag{3_2.63}$$

The cross product of two vectors is

$$\mathbf{v} \times \mathbf{w} = v^r w^s \mathbf{g}_r \times \mathbf{g}_s = v_r w_s \mathbf{g}^r \times \mathbf{g}^s. \tag{3_2.64}$$

Thus, using (3_2.53) we write

$$\mathbf{v} \times \mathbf{w} = \varepsilon_{rst} v^r w^s \mathbf{g}^t = \varepsilon^{rst} v_r w_s \mathbf{g}_t. \tag{3_2.65}$$

From the scalar product of vectors, we have

$$\cos\alpha = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v} \cdot \mathbf{w}|} \tag{3_2.66}$$

where α is the angle between the two vectors, then using (3_2.62) and (2_2.63), we write

$$\cos\alpha = \frac{v^r w^s g_{rs}}{\sqrt{(v^r v^s w^m w^n g_{rs} g_{mn})}}. \tag{3_2.67}$$

If the vectors are perpendicular, then $\cos\alpha = 0$ and (3_2.62) vanishes.

3_2.8 The Christoffel symbols

Let us start by considering the differentiation of equation (3_2.45)

$$g_{s,r} = g_{r,s} = \frac{\partial^2 R}{\partial \vartheta^r \partial \vartheta^s} = R,_{rs} = R,_{sr} \qquad (3\_2.68)$$

and deriving the inverse transformation from (3_2.44) we write

$$i_s = i^s = \frac{\partial x^s}{\partial \vartheta^j} g^j = \frac{\partial \vartheta^j}{\partial x^s} g_j. \qquad (3\_2.69)$$

Taking also the derivatives of (3_2.44), we get

$$g_{r,s} = \frac{\partial^2 x^i}{\partial \vartheta^r \partial \vartheta^s} i_i. \qquad (3\_2.70)$$

Replacing the value of the constant unit vectors in the above expression, we end up with the following

$$g_{r,s} = \Gamma_{rsj} g^j = \Gamma^i_{rs} g_i \qquad (3\_2.71)$$

where

$$\Gamma_{rsj} = \frac{\partial^2 x^i}{\partial \vartheta^r \partial \vartheta^s} \frac{\partial x^i}{\partial \vartheta^j} \quad , \quad \Gamma^i_{rs} = g^{ij} \Gamma_{rsj}. \qquad (3\_2.72)$$

These new symbols are called the Christoffel symbols of the first and second kind respectively. They are also sometimes called the Christoffel three symbols.

The Christoffel symbols of the first kind are also obtained from the derivatives of the metric tensors then, by using (3_2.30) and replacing the functions p and q by x and ϑ respectively

$$g_{lj} = p^i_{,l}\, p^i_{,j} = \frac{\partial x^i}{\partial \vartheta^l} \frac{\partial x^i}{\partial \vartheta^j}$$

$$g^{lj} = q^l_{,i}\, q^j_{,i} = \frac{\partial \vartheta^l}{\partial x^i} \frac{\partial \vartheta^j}{\partial x^i} \tag{3\_2.73}$$

$$g^l_j = q^l_{,i}\, p^i_{,j} = \frac{\partial \vartheta^l}{\partial x^i} \frac{\partial x^i}{\partial \vartheta^j}.$$

Differentiating the first equation of (3_2.73) and interchanging the indices, since the order of differentiation is immaterial, we write the following

$$g_{rs,j} = \frac{\partial^2 x^i}{\partial \vartheta^r \partial \vartheta^j} \frac{\partial x^i}{\partial \vartheta^s} + \frac{\partial x^i}{\partial \vartheta^r} \frac{\partial^2 x^i}{\partial \vartheta^s \partial \vartheta^j}$$

$$g_{js,r} = \frac{\partial^2 x^i}{\partial \vartheta^j \partial \vartheta^r} \frac{\partial x^i}{\partial \vartheta^s} + \frac{\partial x^i}{\partial \vartheta^j} \frac{\partial^2 x^i}{\partial \vartheta^s \partial \vartheta^r} \tag{3\_2.74}$$

$$g_{rj,s} = \frac{\partial^2 x^i}{\partial \vartheta^r \partial \vartheta^s} \frac{\partial x^i}{\partial \vartheta^j} + \frac{\partial x^i}{\partial \vartheta^r} \frac{\partial^2 x^i}{\partial \vartheta^j \partial \vartheta^s}.$$

Therefore, we get by summation of the first two expressions and subtraction of the third

$$g_{rs,j} + g_{js,r} - g_{rj,s} = 2\, \frac{\partial^2 x^i}{\partial \vartheta^j \partial \vartheta^r} \frac{\partial x^i}{\partial \vartheta^s} \tag{3\_2.75}$$

which, in comparison with the first equation of (3_2.72) is

$$\Gamma_{jrs} = \frac{1}{2} \left[g_{rs,j} + g_{js,r} - g_{rj,s} \right]. \tag{3\_2.76}$$

Now, using the second equation of (3_2.72), we write

$$\Gamma^i_{jr} = g^{is} \Gamma_{jrs} = \frac{1}{2} g^{is} \left[g_{rs,j} + g_{js,r} - g_{rj,s} \right]. \tag{3\_2.77}$$

In Cartesian coordinates where the lines are straight and the surfaces flat, the $g_{ij} = \delta^i_j$ are constants and the Christoffel symbols vanish identically. But in general curvilinear coordinates they do not vanish. Thus, the Christoffel symbols are not tensors. Since the metrics are all symmetric then, the Christoffel symbol of the second kind is symmetric with respect to the lower indices.

In the same manner as in (3_2.70) the use of the derivative of the contravariant base vector leads to

$$g^i{}_{,j} = - \Gamma^i_{jr} g^r. \tag{3\_2.78}$$

Then, using (3_2.71) we write the following results

$$\begin{aligned} \Gamma_{irs} &= \Gamma_{ris} = g_s \cdot g_{i,r} = g_s \cdot g_{r,i} \\ \Gamma^s_{ir} &= \Gamma^s_{ri} = g^s \cdot g_{i,r} = g^s \cdot g_{r,i} = - g_i \cdot g^s{}_{,r} \end{aligned} \tag{3\_2.79}$$

and the sum

$$\Gamma_{jrs} + \Gamma_{srj} = g_{js,r}. \tag{3\_2.80}$$

Also use of (3_2.77) gives

$$\Gamma^i_{ir} = \frac{1}{2} g^{is} \left[g_{rs,i} + g_{is,r} - g_{ir,s} \right].$$

With the help of (3_2.74), (3_2.27) we find

$$\begin{aligned} \Gamma^i_{ir} &= \frac{1}{2} g^{is} g_{is,r} \\ &= \frac{1}{2g} \frac{\partial g}{\partial g_{is}} \frac{\partial g_{is}}{\partial \vartheta^r} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial \vartheta^r} . \end{aligned} \tag{3\_2.81}$$

3_2.9 Covariant differentiation

The previous algebra and differentiations were applied to scalars, vectors and tensors specifically at one and the same point of well defined space. In addition to the scalars, vectors and tensors to be isolated, they also appear in fields. In differential geometry, Lass (1950), Bickley (1962) and Wrede (1963), have reported that the processes of differentiation and partial differentiation do not conserve the tensorial character of the field. For instance, we have to mention that the derivative of a covariant vector i.e. tensor of the first order, is not a tensor. Hence, the covariant differentiation appears to be necessary in order to preserve the tensor character of the notation.

In equation (3_2.9), we showed that the derivative of a scalar transforms according to the covariant type of transformation, and it gives a covariant tensor

$$\overset{*}{F}_{,i} = F_{,j}\, q^{j}_{,i}.$$

If $\mathbf{v}$ is a vector, then according to (3_2.9)

$$\overset{*}{\mathbf{v}}_{,i} = \mathbf{v}_{,j}\, q^{j}_{,i} \qquad (3\_2.82)$$

i.e. $\mathbf{v}_{,i} = \dfrac{\partial \mathbf{v}}{\partial \vartheta^{i}}$ transforms according to the covariant rule of transformation. In (3_2.59), we defined $\mathbf{v}$ as

$$\mathbf{v} = v^{r}\, \mathbf{g}_{r} = v_{r}\, \mathbf{g}^{r}$$

then $$\mathbf{v}_{,j} = v^r_{,j}\, \mathbf{g}_r + v^r\, \mathbf{g}_{r,j}$$

$$= v_{r,j}\, \mathbf{g}^r + v_r\, \mathbf{g}^r_{,j}. \tag{3_2.83}$$

Using (3_2.71) and (3_2.78), we write

$$\mathbf{v}_{,j} = v^r|_j\, \mathbf{g}_r = v_r|_j\, \mathbf{g}^r \tag{3_2.84}$$

where the new expressions are set equal to

$$\left.\begin{aligned} v^r|_j &= v^r_{,j} + \Gamma^r_{ij}\, v^i \\ v_r|_j &= v_{r,j} - \Gamma^i_{rj}\, v_i \end{aligned}\right\}. \tag{3_2.85}$$

$v^r|_j$ and $v_r|_j$ are the covariant derivatives of the contravariant and covariant components of the vector **v**. The derivatives of these components form a tensor of order two. In a similar way, we can write the covariant derivatives of a tensor of order two as

$$\left.\begin{aligned} A_{ij}|_k &= A_{ij,k} - \Gamma^s_{ik}\, A_{sj} - \Gamma^s_{jk}\, A_{is} \\ A^{i}_{\cdot j}|_k &= A^{i}_{\cdot j,s} + \Gamma^i_{ks}\, A^{s}_{\cdot j} - \Gamma^s_{jk}\, A^{i}_{\cdot s} \\ A^{ij}|_k &= A^{ij}_{,k} + \Gamma^i_{ks}\, A^{sj} + \Gamma^j_{ks}\, A^{is} \end{aligned}\right\}. \tag{3_2.86}$$

Since the covariant derivatives are tensors, then their components are raised and lowered following the rules given in (3_2.34) to (3_2.38), the result will be called contravariant differentiation, then

$$\left.\begin{aligned} A_{ij}|^s &= g^{sn}\, A_{ij}|_n \\ A^{ij}|^s &= g^{sn}\, A^{ij}|_n \end{aligned}\right\}. \tag{3_2.87}$$

A very special result to be noted here is that when the metrics are constants, as in the orthogonal Cartesian coordinates, we find that the Christoffel symbols are all zero and the covariant derivatives of the metrics vanish in this coordinates. Since the covariant derivatives are themselves tensors, then if they vanish in one system of coordinates they vanish in all and, we find

$$\left.\begin{aligned} g_{ij}\big|_s = g^{ij}\big|_s = g^i_j\big|_s = \delta^i_j\big|_s = 0 \\ g^{ij}\big|^s = 0 \end{aligned}\right\}. \qquad (3\_2.88)$$

For similar reason, the covariant derivatives of the permutation symbols are all zero then, we write

$$\varepsilon_{rst}\big|_j = 0, \qquad \varepsilon^{rst}\big|_j = 0 \ . \qquad (3\_2.89)$$

The covariant derivatives can be generalized to tensors of higher order where always in the result, the first term is the partial derivative. Then, for each contravariant and covariant index, a term with a Christoffel symbol is added and a term subtracted respectively. The order of the result is always one covariant order higher, thus

$$\left.\begin{aligned} A^{j..k}_{.q..r}\big|_p = A^{j..k}_{.q..r,p} + \Gamma^j_{ip} A^{i..k}_{.q..r} + \ldots + \Gamma^k_{ip} A^{j..i}_{.q..r} \\ - \Gamma^i_{qp} A^{j..k}_{.i..r} - \ldots - \Gamma^i_{rp} A^{j..k}_{.q..i} \end{aligned}\right\}. \qquad (3\_2.90)$$

3_2.10 The Riemann_Christoffel tensor

It appears now systematic to find the covariant differentiation of any tensor of any order since the general rule is established in equation (3_2.90). Equations (3_2.85) shows that the result of a covariant differentiation is a term containing the partial derivative plus the change of the base vector as given by the Christoffel symbol. Therefore, if one wants to compute the covariant derivative of covariantly differentiated vector v_r, then

$$(v_r|_j)|_i = v_r|_{ji} = (v_{r,j} - \Gamma^s_{rj}\, v_s)|_i$$

$$= (v_{r,j} - \Gamma^s_{rj}\, v_s)_{,i} - \Gamma^k_{ri}(v_{j,k} - \Gamma^s_{jk}\, v_s) - \Gamma^k_{ji}(v_{k,r} - \Gamma^s_{kr}\, v_s)$$

$$= v_{r,ji} - \Gamma^s_{rj,i} v_s - \Gamma^s_{rj}\, v_{s,i} - \Gamma^k_{ri} v_{j,k} + \Gamma^k_{ri}\Gamma^s_{jk}\, v_s - \Gamma^k_{ji} v_{k,r} + \Gamma^k_{ji}\Gamma^s_{kr}\, v_s$$

$(v_r|_j)|_i$ is then called, the second covariant derivative of the covariant vector v_r. If now, we consider the difference of two twice covariantly differentiated covariant vectors with permuted indices, then after some calculation we find

$$v_i|_{jk} - v_i|_{kj} = \left[\Gamma^n_{ki}\Gamma^m_{nj} - \Gamma^n_{kj}\Gamma^m_{ni} + \Gamma^m_{ki,j} - \Gamma^m_{kj,i} \right] v_m. \tag{3_2.91}$$

Then, according to the rigorous quotient theorem, since v_m is arbitrary, the expression between brackets is a tensor

$$v_i|_{jk} - v_i|_{kj} = R^m{}_{.ijk} v_m. \tag{3_2.92}$$

The new expression $R^m{}_{.ijk}$ is called the Riemann_Christoffel tensor and is of order four. Inspection of equation (3_2.91) shows that this tensor consists of the components of the metric tensor and their derivatives up to the second order. Therefore the Riemann-Christoffel vanish identically in a Cartesian coordinate system. By lowering the index m, we write

$$R_{pijk} = g_{mp} \, R^{m}{}_{.ijk}. \qquad (3\_2.93)$$

The Riemann_Christoffel tensor can be written using the metrics and their derivatives in the following manner

$$R_{pijk} = \frac{1}{2}(g_{pk,ij} + g_{ij,pk} - g_{pj,ik} - g_{ik,pj}) + g^{rs}(\Gamma_{ijr}\,\Gamma_{pks} - \Gamma_{ikr}\Gamma_{pjs})$$

which has the characteristic of being skew_symmetric in pi and jk, then we write

$$R_{pijk} = -\,R_{ipjk} \;,\; R_{pijk} = -\,R_{pikj} \;,\; R_{pijk} = R_{jkpi}.$$

That is to say only, six out of 81 components are independent and every component of the rest is either zero or equal to plus or minus one of the six components. These six components are

$$R_{3131}\,, R_{3232}\,, R_{1212}\,, R_{3132}, R_{3212}, R_{3112}.$$

If we define the Euclidean space as the space in which a Cartesian coordinates system can be adopted. Then, the Riemann-Christoffel vanish in all other coordinate systems which can be established in the Euclidean space. That is to say

$$R^{m}{}_{.ijk} = 0. \qquad (3\_2.94)$$

The second covariant derivatives of a tensor of order two are

$$T_{ij}|_{rs} - T_{ij}|_{sr} = T_{nj}\, R^{n}_{.\,irs} + T_{in}\, R^{n}_{.\,jrs}$$

$$T^{ij}|_{rs} - T^{ij}|_{sr} = -\, T^{nj}\, R^{i}_{.\,nrs} - T^{in}\, R^{j}_{.\,nrs}\,. \tag{3_2.95}$$

Also, from (3_2.92) the order of covariant differentiation in Euclidean space is immaterial and we find

$$\left.\begin{aligned} v_i|_{jk} &= v_i|_{kj} \\ v_{is}|_{jk} &= v_{is}|_{kj} \\ v^{is}|_{jk} &= v^{is}|_{kj} \end{aligned}\right\}. \tag{3_2.96}$$

3_3 Geometry of surface

A point which is at a distance ϑ^3 from the reference surface of a shell, fig.(3_3.1), has the position vector, (Green & Zerna (1968))

$$\mathbf{R} = \mathbf{r} + \vartheta^3 \mathbf{a}_3. \tag{3_3.1}$$

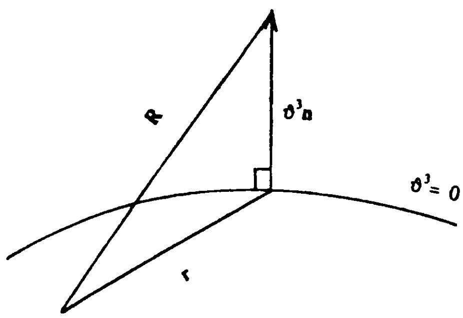


fig.(3_3.1) the position vector of a surface

In equation (3_3.1) $\mathbf{r}$ is function of $(\vartheta^1, \vartheta^2)$ and $\mathbf{a}_3$ is a vector of unit magnitude which also depends on $(\vartheta^1, \vartheta^2)$. The equation $\vartheta^3 = 0$ determines the reference surface. The vector $\mathbf{a}_3$ is perpendicular to the reference surface and is called the normal vector. Equation (3_3.1) for the surface becomes

$$\mathbf{r} = \mathbf{r}(\vartheta^1, \vartheta^2) \tag{3\_3.2}$$

If one of the coordinates $(\vartheta^1, \vartheta^2)$ is kept constant, then (3_3.2) describes a curve that lies wholly on the surface and as the constant is varied, we end up with a family of curves. If the same process is repeated with the second coordinate, another family of curves is obtained.

The two families of curves constitute on the surface a system of curvilinear coordinates. Lines of constant ϑ^1 or ϑ^2 can be drawn on the surface and in general they will not cross at right angles nor will there be a constant spacing between intersections.

At each point on the surface, there are two sets of base vectors. Firstly, there are the covariant base vectors which, by analogy to (3_2.45), are given by

$$\mathbf{a}_\alpha = \mathbf{r},_\alpha = \frac{\partial \mathbf{r}}{\partial \vartheta^\alpha}. \tag{3\_3.3}$$

In (3_3.3) and in all that follows the Greek indices will range over the values 1 and 2. Thus (3_3.3) defines two base vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, they lie in the local plane of the surface in the directions of the curvilinear coordinates, in general they are not unit vectors. Also (3_3.3) shows that $\mathbf{a}_\alpha$ transforms according to the covariant rule for the surface transformation of coordinates.

The second set of base vectors are the contravariant base vectors $\mathbf{a}^1$ and $\mathbf{a}^2$. According to (3_2.47) these base vectors lie also in the local plane of the surface and have directions and magnitude such that the scalar product

$$\mathbf{a}^\alpha . \mathbf{a}_\beta = \delta^\alpha_\beta \tag{3\_3.4}$$

where δ^α_β is the Kronecker delta for the surface and is defined by

$$\delta^\alpha_\beta = \begin{cases} = 1 & \text{if } \alpha = \beta \\ = 0 & \text{if } \alpha \neq \beta \end{cases} \tag{3\_3.5}$$

From (3_3.4) $\mathbf{a}^1$ is perpendicular to $\mathbf{a}_2$. Similarly $\mathbf{a}^2$ is perpendicular to $\mathbf{a}_1$. The third base vector $\mathbf{a}_3 = \mathbf{a}^3$, is the local unit normal to the surface. It is perpendicular to both base vectors $\mathbf{a}_\alpha$ and $\mathbf{a}^\alpha$, so that

$$\mathbf{a}_3 = \frac{\left(\mathbf{a}_1 \times \mathbf{a}_2 \right)}{| \mathbf{a}_1 \times \mathbf{a}_2 |} . \tag{3\_3.6}$$

We conclude from the previous definitions that

$$\left. \begin{aligned} \mathbf{a}_3 \cdot \mathbf{a}_\alpha &= 0 \\ \mathbf{a}_3 \cdot \mathbf{a}_3 &= 1 \\ \mathbf{a}_3 \cdot \mathbf{a}_{3,\alpha} &= 0 \end{aligned} \right\} . \tag{3\_3.7}$$

Using again (3_2.47), with $\vartheta^3 = 0$ the symmetric metric surface tensors are

$$\mathbf{a}_\alpha \cdot \mathbf{a}_\beta = a_{\alpha\beta} \quad , \quad a_{\alpha 3} = a_{3\alpha} = 0 \quad , \quad a_{33} = 1. \tag{3\_3.8}$$

From equation (3_2.29) with $\vartheta^3 = 0$ we have

$$a_{\alpha\beta} a^{\beta\rho} = \delta^\rho_\alpha \quad , \quad a^{\alpha 3} = 0 \quad , \quad a^{33} = 1. \tag{3\_3.9}$$

Hence

$$a^{11} = \frac{a_{22}}{a} \; , \; a^{12} = a^{21} = -\frac{a_{12}}{a} \; , \; a^{22} = \frac{a_{11}}{a}. \tag{3\_3.10}$$

The quantity, a, is defined as

$$|a_{\alpha\beta}| = a = a_{11} a_{22} - (a_{12})^2. \tag{3\_3.11}$$

Equations (3_3.8) and (3_3.9) give the covariant, contravariant and mixed metric surface tensors, they are second order tensors in that they each have two indices. All these tensors satisfy the appropriate laws for transformations of surface coordinates. By setting $\vartheta^3 = 0$ in equation (3_2.48) it follows that the base vectors can be expressed in terms of each other as

$$\mathbf{a}_\alpha = a_{\alpha\beta}\, \mathbf{a}^\beta \quad , \quad \mathbf{a}^\alpha = a^{\alpha\beta}\, \mathbf{a}_\beta. \tag{3\_3.12}$$

The magnitudes of the covariant and contravariant base vectors are from (3_2.51)

$$|\mathbf{a}_\alpha| = \sqrt{(\mathbf{a}_\alpha \cdot \mathbf{a}_\alpha)} = \sqrt{a_{\alpha\alpha}}$$

$$(\alpha \text{ not summed})$$

$$|\mathbf{a}^\alpha| = \sqrt{(\mathbf{a}^\alpha \cdot \mathbf{a}^\alpha)} = \sqrt{a^{\alpha\alpha}} \tag{3\_3.13}$$

3_3.1 First fundamental form

Using equation (3_3.3) we write

$$dr = a_{\alpha}\, d\vartheta^{\alpha}. \qquad (3\_3.14)$$

Then, equation (3_2.49) expressing the line element becomes for the surface

$$ds^2 = dr.dr = a_{\alpha\beta}\, d\vartheta^{\alpha}\, d\vartheta^{\beta}. \qquad (3\_3.15)$$

Equation (3_3.15) is known as the **first fundamental form** of the surface. It is a quadratic differential form, where its square root ds represents the distance between two adjacent points with coordinates (ϑ^1, ϑ^2) and ($\vartheta^1 + d\vartheta^1$, $\vartheta^2 + d\vartheta^2$) on the surface. Since ds is length, then is always positive (except when both $d\vartheta^1 = d\vartheta^2 = 0$) when concerned with real surfaces.

The line element along the coordinate curves is given from (3_2.50) by

$$ds_{\alpha} = a_{\alpha}\, d\vartheta^{\alpha} \qquad \left(\alpha \text{ not summed}\right) \qquad (3\_3.16)$$

where the magnitude of the line element is

$$ds_{\alpha} = \sqrt{a_{\alpha\alpha}}\, d\vartheta^{\alpha}. \qquad (3\_3.17)$$

The angle γ between the coordinate curves of the surface is given by the following formula

$$\cos\gamma = \frac{ds_1 . ds_2}{ds_1 \; ds_2} = \frac{a_{12}}{\sqrt{(a_{11}a_{22})}} . \qquad (3\_3.18)$$

If the curves are orthogonal, then

$$\cos\gamma = 0 \quad , \quad a_{12} = a^{12} = 0. \qquad (3\_3.19)$$

The area of the surface bounded by the two coordinate lines passing through $(\vartheta^1, \vartheta^2)$ and the coordinate lines passing through $(\vartheta^1 + d\vartheta^1, \vartheta^2 + d\vartheta^2)$ is from (3_2.57) by putting $\vartheta^3 = 0$,

$$dS = \sqrt{a} \; d\vartheta^1 d\vartheta^2. \qquad (3\_3.20)$$

3_3.2 Second fundamental form

The application of the ε_system to the surface is defined by $\varepsilon^{\alpha\beta 3}$ and $\varepsilon_{\alpha\beta 3}$, it becomes $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$ when the third coordinate component ϑ^3 is equal to zero, hence we write

$$\begin{aligned} &\varepsilon_{12} = -\varepsilon_{21} = \sqrt{a} \quad , \quad \varepsilon^{12} = -\varepsilon^{21} = 1/\sqrt{a} \\ &\varepsilon_{11} = \varepsilon_{22} = \varepsilon^{11} = \varepsilon^{22} = 0. \end{aligned} \qquad (3\_3.21)$$

We add also the following formulae

$$\begin{aligned} \varepsilon^{\alpha\beta} &= a^{\alpha\gamma} a^{\beta\rho} \varepsilon_{\gamma\rho} \\ \varepsilon_{\alpha\beta} &= a_{\alpha\gamma} a_{\beta\rho} \varepsilon^{\gamma\rho}. \end{aligned} \qquad (3\_3.22)$$

The base vector products, when $\vartheta^3 = 0$, are given from (3_2.53) by

$$\begin{aligned} \mathbf{a}_\alpha \times \mathbf{a}_\beta &= \varepsilon_{\alpha\beta}\, \mathbf{a}_3 \\ \mathbf{a}^\alpha \times \mathbf{a}^\beta &= \varepsilon^{\alpha\beta}\, \mathbf{a}_3 \\ \mathbf{a}_3 \times \mathbf{a}_\beta &= \varepsilon_{\beta\gamma}\, \mathbf{a}^\gamma \\ \mathbf{a}_3 \times \mathbf{a}^\beta &= \varepsilon^{\beta\gamma}\, \mathbf{a}_\gamma \end{aligned} \qquad (3\_3.23)$$

From (3_2.54) the scalars triple product of the base vectors are

$$\left.\begin{aligned} \left[\mathbf{a}_\alpha\ \mathbf{a}_\beta\ \mathbf{a}_3\right] &= \varepsilon_{\alpha\beta} \\ \left[\mathbf{a}^\alpha\ \mathbf{a}^\beta\ \mathbf{a}^3\right] &= \varepsilon^{\alpha\beta} \end{aligned}\right\}. \qquad (3\_3.24)$$

Now, the scalar product

$$\mathbf{dr} \,.\, \mathbf{da}_3 = -\, b_{\alpha\beta}\, d\vartheta^\alpha d\vartheta^\beta \qquad (3\_3.25)$$

where $b_{\alpha\beta}$ is given by

$$b_{\alpha\beta} = b_{\beta\alpha} = \mathbf{a}_3 . \frac{\partial^2 \mathbf{r}}{\partial\vartheta^\alpha \partial\vartheta^\beta}$$

$$= \mathbf{a}_3 .\, \mathbf{r}_{,\alpha\beta} = \mathbf{a}_3 . \mathbf{a}_{\alpha,\beta} = -\, \mathbf{a}_\alpha . \mathbf{a}_{3,\beta} = -\, \mathbf{a}_\beta . \mathbf{a}_{3,\alpha} . \qquad (3\_3.26)$$

Equation (3_3.25) is known as the **second fundamental form** of the surface. $b_{\alpha\beta}$ and $b^{\alpha\beta}$ are symmetric surface tensors of order two and are related as follows

$$\begin{aligned} b^\alpha_\beta &= a^{\alpha\gamma} b_{\beta\gamma} = a_{\beta\gamma} b^{\alpha\gamma} \\ b^{\alpha\beta} &= a^{\alpha\gamma} b^\beta_\gamma \\ b_{\alpha\beta} &= a_{\alpha\gamma} b^\gamma_\beta . \end{aligned} \qquad (3\_3.27)$$

According to Struik (1961), there are no (real) directions for which the first fundamental form is zero, whereas it may happen that there are real directions for which the second fundamental form is zero, i.e

$$dr \, . \, da_3 = - b_{\alpha\beta} \, d\vartheta^{\alpha} d\vartheta^{\beta} = 0. \qquad (3\_3.28)$$

The directions which satisfy the above equation are called asymptotic directions and curves having these directions are called asymptotic curves. The normal a_3, is a unit vector normal to the surface and therefore da_3 lies in the plane of the surface.

In geometry, see Hilbert & Cohn_Vossen (1952), one of the classifications of surfaces is related to characteristics or alternatively called asymptotic lines on the surface. In hyperbolic surfaces, as the hyperboloid of one sheet or the catenoid, there are two asymptotic lines. there is one asymptotic line on surfaces of parabolic types, as the cylinder, whereas there are no real asymptotic lines on the elliptic surfaces such as the sphere and the ellipsoid.

Consider now a small change in the unit normal to the surface given by analogy to dr in the following manner

$$da_3 = a_{3,\alpha} d\vartheta^{\alpha} = - b^{\lambda}_{\alpha} \, a_{\lambda} d\vartheta^{\alpha}. \qquad (3\_3.29)$$

Then, we can resolve da_3 into two components

$$da_3 = k_n \, dr + \tau \, (dr \times a_3) \qquad (3\_3.30)$$

$(dr \times a_3)$ lies in the plane of the surface and has magnitude equal to that of dr. The scalar products of the above equation with **dr** and then with $(dr \times a_3)$ give the **normal curvature** and the **twist** of the surface respectively. From (3_3.28), (3_3.29) and (3_3.30)

$$k_n = - \frac{(dr \;.\; da_3)}{(dr \;.\; dr\;)} = \frac{b_{\alpha\beta}\; d\vartheta^{\alpha} d\vartheta^{\beta}}{a_{\gamma\eta}\; d\vartheta^{\gamma} d\vartheta^{\eta}} = \frac{\text{the Sec.Fun.For}}{\text{the Fir.Fun.For}}$$

$$\tau = \frac{da_3 . (dr \times a_3)}{(dr \times a_3) . (dr \times a_3)} = \frac{da_3 . (dr \times a_3)}{(\; dr \;.\; dr\;)} = \frac{b^{\beta}_{\alpha}\; \varepsilon_{\gamma\beta}\; d\vartheta^{\alpha} d\vartheta^{\gamma}}{a_{\lambda\eta}\; d\vartheta^{\lambda} d\vartheta^{\eta}}. \qquad (3\_3.31)$$

When the second fundamental form of the surface vanishes, the normal curvature is zero, therefore the asymptotic curves have zero normal curvature.

Now, the vectors da_3 and dr are parallel if

$$da_3 + k\; dr = 0 \qquad (3\_3.32)$$

which is called Rodrigues's formula, and k is a scalar. Using (3_3.14) and (3_3.29), then (3_3.32) becomes

$$\left(b^{\lambda}_{\beta}\; a_{\lambda} - k\; a_{\beta} \right) d\vartheta^{\beta} = 0. \qquad (3\_3.33)$$

Multiplication of the above equation by a^{α} yields

$$\left(b^{\alpha}_{\beta} - k\; \delta^{\alpha}_{\beta} \right) d\vartheta^{\beta} = 0. \qquad (3\_3.34)$$

In texts on differential geometry, see Coxeter (1961), equation (2_3.34) provides useful information concerning the principal curvatures and the principal directions on the surface. On performing the summation implied in (3_3.34) by the repeated β and noting that α range over the value 1 and 2

$$\left.\begin{aligned}(b^1_1 - k)\, d\vartheta^1 + b^1_2\, d\vartheta^2 &= 0\\ b^2_1 d\vartheta^1 + (b^2_2 - k)\, d\vartheta^2 &= 0\end{aligned}\right\}$$

which has a non_trivial solution $d\vartheta^\beta \neq 0$, if

$$\begin{vmatrix} b^1_1 - k & b^1_2 \\ b^2_1 & b^2_2 - k \end{vmatrix} = 0.$$

Then,

$$k^2 - b^\alpha_\alpha\, k + \det(b^\beta_\alpha) = 0 \qquad (3\_3.35)$$

which is a quadratic equation for k, the corresponding two values of the ratio $d\vartheta^2/\, d\vartheta^1$ give the minimum and the maximum values of the curvature. The two values of k are the principal curvatures and they always occur in orthogonal directions (unless the two values of k are equal in which case all directions are principal directions). The product and the arithmetic mean of these principal curvatures are the **Gaussian curvature** K and the **mean curvature** H, thus (3_3.35) becomes

$$k^2 - 2\, H\, k + K = 0.$$

The principal curvatures are

$$\begin{aligned} k_1 &= H + \sqrt{(H^2 - K)} \\ k_2 &= H - \sqrt{(H^2 - K)} \end{aligned} \qquad (3\_3.36)$$

where

$$2\, H = k_1 + k_2 = b^\alpha_\alpha \qquad (3\_3.37)$$

$$K = k_1 \; k_2 = \det(b_\alpha^\beta) = b_1^1 \; b_2^2 - b_2^1 \; b_1^2. \qquad (3\_3.38)$$

Equation (3_3.38) can also be written as

$$K = b_1^1 \; b_2^2 - b_2^1 \; b_1^2 = \frac{\left(b_{11} b_{22} - (b_{12})^2 \right)}{\left(a_{11} a_{22} - (a_{12})^2 \right)} = \frac{| \; b_{\alpha\beta} |}{a} \qquad (3\_3.39)$$

which is the ratio of the two fundamental determinants.

Investigation of (3_3.39) shows that three cases of K have to be distinguished. First, when K is positive, the normal curvature has the same sign in all directions and hence the surface is called synclastic. Examples of these surfaces are, the ellipsoids and elliptic paraboloids. Secondly, is the case of negative K, where the normal curvature changes sign on the surface twice before the normal plane to the surface accomplish half turn about its axis of rotation. Therefore the normal curvature is zero in the two directions which we have called already the asymptotic directions. Examples of these surfaces are called anticlastic or saddle-shaped surfaces, among which are the catenoid and the hyperboloid of one sheet. A single surface may be synclastic in some regions and anticlastic in some others. Examples of these surfaces are the bell shaped shells and the torus. The regions on these surfaces are separated by a locus of parabolic points, where K = 0. Lastly, surfaces where K is zero are called developable. In these surfaces, one or both of the principal curvatures is zero. If one principal curvature is zero, this will be in the only asymptotic direction and if both principal curvatures are zero, the surface is a plane without normal curvature. Examples of these, are respectively the cylinder and the plane.

3_3.3 Christoffel symbols

The Christoffel symbols of the first kind with respect to the surface $\vartheta^3 = 0$ are

$$\Gamma_{\beta\gamma\alpha} = \frac{1}{2}\left(a_{\alpha\beta,\gamma} + a_{\alpha\gamma,\beta} - a_{\beta\gamma,\alpha} \right) \tag{3\_3.40}$$

and the Christoffel symbols of the second kind are

$$\Gamma^{\alpha}_{\beta\gamma} = a^{\alpha\lambda}\, \Gamma_{\beta\gamma\lambda} = \mathbf{a}^{\alpha} . \mathbf{a}_{\gamma,\beta} = \mathbf{a}^{\alpha} . \mathbf{a}_{\beta,\gamma} = - \mathbf{a}_{\gamma} \mathbf{a}^{\alpha}{}_{,\beta}$$

$$\Gamma^{\alpha}_{\beta 3} = \mathbf{a}^{\alpha} . \mathbf{a}_{3,\beta} = - \mathbf{a}_{3} . \mathbf{a}^{\alpha}{}_{,\beta} = - b^{\alpha}_{\beta}$$

$$\Gamma^{3}_{\alpha\beta} = \mathbf{a}^{3} . \mathbf{a}_{\alpha,\beta} = - \mathbf{a}_{\beta} . \mathbf{a}^{3}{}_{,\beta} = b_{\alpha\beta}$$

$$\Gamma^{3}_{\alpha 3} = \mathbf{a}^{3} . \mathbf{a}_{3,\alpha} = 0$$

$$\Gamma^{\lambda}_{\lambda\alpha} = \frac{1}{\sqrt{a}} \frac{\partial \sqrt{a}}{\partial \vartheta^{\alpha}}$$

$$\Gamma^{3}_{33} = 0. \tag{3\_3.41}$$

$\Gamma^{\alpha}_{\beta\gamma}$ can be expressed also in terms of the metric surface tensor as follows

$$\Gamma^{\lambda}_{\alpha\beta} = a^{\lambda\rho} \mathbf{a}_{\rho} . \mathbf{a}_{\alpha,\beta} . \tag{3\_3.42}$$

The Riemann_Christoffel tensor for the surface is obtained by putting $\vartheta^3 = 0$ in the combined two equations (3_2.91) and (3_2.92), thus

$$R^{\rho}{}_{\cdot\alpha\beta\psi} = \left[\Gamma^{\gamma}_{\psi\alpha}\Gamma^{\rho}_{\gamma\beta} - \Gamma^{\gamma}_{\psi\beta}\Gamma^{\rho}_{\gamma\alpha} + \Gamma^{\rho}_{\psi\alpha,\beta} - \Gamma^{\rho}_{\psi\beta,\alpha}\right] + \left[\Gamma^{3}_{\psi\alpha}\, \Gamma^{\rho}_{3\beta} - \Gamma^{3}_{\psi\beta}\, \Gamma^{\rho}_{3\alpha}\right]$$

$$R^{\rho}{}_{\cdot\alpha\beta\psi} = \overline{R^{\rho}}{}_{\cdot\alpha\beta\psi} + \left[\Gamma^{3}_{\psi\alpha}\, \Gamma^{\rho}_{3\beta} - \Gamma^{3}_{\psi\beta}\, \Gamma^{\rho}_{3\alpha}\right].$$

From (3_2.94), we have

$$R^{\rho}{}_{\cdot\alpha\beta\psi} = 0.$$

Then the previous equation becomes

$$\overline{R^{\rho}}{}_{\cdot\alpha\beta\psi} = - \left[\Gamma^{3}_{\psi\alpha}\, \Gamma^{\rho}_{3\beta} - \Gamma^{3}_{\psi\beta}\, \Gamma^{\rho}_{3\alpha}\right].$$

In what follows the bar over the Riemann symbol will be omitted, and therefore the Riemann_Christoffel tensor for the surface is

$$R^{\rho}{}_{\cdot\alpha\beta\psi} = \Gamma^{3}_{\alpha\beta}\, \Gamma^{\rho}_{3\psi} - \Gamma^{3}_{\alpha\psi}\, \Gamma^{\rho}_{3\beta} = b_{\alpha\beta}(-b^{\rho}_{\psi}) - b_{\alpha\psi}(-b^{\rho}_{\psi}). \qquad (3\_3.43)$$

Lowering the index, we get

$$R_{\rho\alpha\beta\psi} = a_{\rho\gamma}\, R^{\gamma}{}_{\cdot\alpha\beta\psi} = b_{\alpha\psi}b_{\rho\beta} - b_{\alpha\beta}b_{\rho\psi} \qquad (3\_3.44)$$

where from the symmetry of $b_{\alpha\beta}$ we have

$$R_{\alpha\alpha\beta\psi} = R_{\rho\alpha\beta\beta} = 0 \qquad (\alpha,\beta \text{ not summed})$$

$$R_{1212} = R_{2121} = - R_{2112} = - R_{1221}. \qquad (3\_3.45)$$

Thus from (3_3.44) and (3_3.39), we get

$$K = \frac{R_{1212}}{a} = \frac{|b_{\alpha\beta}|}{|a_{\alpha\beta}|} \qquad (3\_3.46)$$

where

$$R_{1212} = |b_{\alpha\beta}| = b_{11}b_{22} - (b_{12})^2. \tag{3_3.47}$$

Also, we can write

$$\begin{aligned} K &= 1/4\ \varepsilon^{\alpha\beta}\varepsilon^{\lambda\nu}\ R_{\alpha\beta\lambda\nu} \\ K &= 1/2\ \varepsilon^{\alpha\beta}\varepsilon^{\lambda\nu}\ b_{\alpha\lambda}b_{\beta\nu} \end{aligned} \tag{3_3.48}$$

By analogy to (3_2.85), if $\vartheta^3 = 0$ then, the covariant differentiation rule for the surface is as follows

$$\left.\begin{aligned} v^{\alpha}|_{\beta} &= v^{\alpha}_{,\beta} + \Gamma^{\alpha}_{\rho\beta}\ v^{\rho} \\ v_{\alpha}|_{\beta} &= v_{\alpha,\beta} - \Gamma^{\rho}_{\alpha\beta}\ v_{\rho} \end{aligned}\right\}. \tag{3_3.49}$$

This rule can be extended to tensors of higher order, then from (3_2.86)

$$\left.\begin{aligned} A_{\alpha\beta}|_{\gamma} &= A_{\alpha\beta,\gamma} - \Gamma^{\rho}_{\alpha\gamma}\ A_{\rho\beta} - \Gamma^{\rho}_{\beta\gamma}\ A_{\alpha\rho} \\ A^{\alpha}_{\cdot\beta}|_{\gamma} &= A^{\alpha}_{\cdot\beta,\gamma} + \Gamma^{\alpha}_{\gamma\rho}\ A^{\rho}_{\cdot\beta} - \Gamma^{\rho}_{\beta\gamma}\ A^{\alpha}_{\cdot\rho} \\ A^{\alpha\beta}|_{\gamma} &= A^{\alpha\beta}_{,\gamma} + \Gamma^{\alpha}_{\gamma\rho}\ A^{\rho\beta} + \Gamma^{\beta}_{\gamma\rho}\ A^{\alpha\rho} \end{aligned}\right\}. \tag{3_3.50}$$

Now, combining equations (3_3.26) and the first equation of (3_3.41) we get **Weingarten and Gauss formulae**

$$\begin{aligned} a_{\alpha,\beta} &= b_{\alpha\beta}\ a_3 + \Gamma^{\lambda}_{\alpha\beta}\ a_{\lambda} \\ a^{\alpha}_{,\beta} &= b^{\alpha}_{\beta}\ a_3 - \Gamma^{\alpha}_{\lambda\beta}\ a^{\lambda} \\ a_{3,\beta} &= -\ b^{\lambda}_{\beta}\ a_{\lambda}. \end{aligned} \tag{3_3.51}$$

The first two equations express the differential character of the relations that link the coefficients of the first fundamental form and those of the second fundamental form and are due to Gauss. However, the third equation is the Weingarten equation and it express the derivatives of the normal in two directions. Also it will be used for definition of the third fundamental form of the surface.

3_3.3.1 Gauss and Codazzi equations

The first equation of (3_3.51) comprises three non independents differential equations defining the coordinates ϑ^{α} of the surface. Setting $\beta = 1$ and differentiating with respect to ϑ^2 gives a similar result to setting $\beta = 2$ and differentiating with respect to ϑ^1, thus

$$\mathbf{a}_{\alpha,12} = b_{\alpha 1,2}\mathbf{a}_3 + b_{\alpha 1}\mathbf{a}_{3,2} + \Gamma^{\lambda}_{\alpha 1,2}\mathbf{a}_{\lambda} + \Gamma^{\lambda}_{\alpha 1}\,\mathbf{a}_{\lambda,2}$$

$$\mathbf{a}_{\alpha,21} = b_{\alpha 2,1}\mathbf{a}_3 + b_{\alpha 2}\mathbf{a}_{3,1} + \Gamma^{\lambda}_{\alpha 2,1}\mathbf{a}_{\lambda} + \Gamma^{\lambda}_{\alpha 2}\,\mathbf{a}_{\lambda,1}.$$

Hence

$$b_{\alpha 1,2}\mathbf{a}_3 + b_{\alpha 1}\mathbf{a}_{3,2} + \Gamma^{\lambda}_{\alpha 1,2}\mathbf{a}_{\lambda} + \Gamma^{\lambda}_{\alpha 1}\,\mathbf{a}_{\lambda,2} = b_{\alpha 2,1}\mathbf{a}_3 + b_{\alpha 2}\mathbf{a}_{3,1} + \Gamma^{\lambda}_{\alpha 2,1}\mathbf{a}_{\lambda} + \Gamma^{\lambda}_{\alpha 2}\,\mathbf{a}_{\lambda,1}. \qquad (3\_3.52)$$

Scalar multiplication by $\mathbf{a}_3$ and using Gauss equation, gives

$$b_{\alpha 1,2} + \Gamma^{\lambda}_{\alpha 1}\,b_{\lambda 2} = b_{\alpha 2,1} + \Gamma^{\lambda}_{\alpha 2}\,b_{\lambda 1}. \qquad (3\_3.53)$$

Using the notion of the covariant differentiation, we thus have

$$b_{\alpha 1}|_2 = b_{\alpha 2}|_1 . \qquad (\ \alpha = 1\ ,2)$$

In general form

$$b_{\alpha\gamma}|_\beta = b_{\alpha\beta}|_\gamma \qquad (3\_3.54)$$

which are known as the **Codazzi equations.**

Again multiplying (3_3.52) by a_γ instead of a_3 we end up with

$$b_{\alpha 1} a_{3,2} . a_\gamma + \Gamma^\lambda_{\alpha 1,2} a_\lambda . a_\gamma + \Gamma^\lambda_{\alpha 1} a_{\lambda,2} . a_\gamma =$$

$$b_{\alpha 2} a_{3,1} . a_\gamma + \Gamma^\lambda_{\alpha 2,1} a_\lambda . a_\gamma + \Gamma^\lambda_{\alpha 2} a_{\lambda,1} . a_\gamma .$$

Using the Weingarten equation and (3_2.79), we write

$$b_{\alpha 2} b_{1\gamma} - b_{\alpha 1} b_{2\gamma} = \left(\Gamma^\lambda_{\alpha 2,1} - \Gamma^\lambda_{\alpha 1,2} \right) a_{\lambda\gamma} + \left(\Gamma^\lambda_{\alpha 2} \Gamma^\rho_{1\gamma} - \Gamma^\lambda_{\alpha 1} \Gamma^\rho_{2\gamma} \right) a_{\lambda\rho}$$

If we set $\gamma = \alpha$ in the above equation, then it vanishes identically. If $\gamma \neq \alpha$ then we write

$$K = \frac{1}{a} \left(a_{12,12} - \frac{1}{2} \left(a_{11,22} + a_{22,11} \right) - \left(\Gamma^\lambda_{12} \Gamma^\rho_{12} - \Gamma^\lambda_{11} \Gamma^\rho_{12} \right) a_{\lambda\rho} \right) . \qquad (3\_3.55)$$

Equation (3_3.55) is called **Gauss' Theorem "Theorema egregium".** Gauss' theorem and the Codazzi equations are the compatibility equations of the surface. An extra fact obtained from Gauss' theorem, is that the Gaussian curvature can be expressed using only the metrics and their derivatives. Any expression which depends only on the coefficients of the first fundamental form is called a bending invariant, hence the Gaussian curvature of the surface is a bending invariant. Therefore, if a given surface is subject to pure bending, we would expect an unchanged Gaussian curvature i.e. distances and angles between points on the surface before pure bending deformation remain the same after it.

It is that in spaces of two dimensions, the intrinsic geometry represented by the coefficients of the first fundamental form is capable of defining some geometric characteristics such as angles, distances and surface areas. However, it is hardly possible for the intrinsic geometry to determine the surface uniquely, as seen from the point of view of embedded surfaces. In this context the second fundamental form of the surface has to be considered and a number of additional conditions have to be satisfied. Among these conditions we state firstly, that the determinant of the first fundamental form of the surface must be positive . Then, both the first and the second fundamental forms have to satisfy the Gauss and Codazzi equations, to ensure the continuity of the surface .

The above statement constitutes the **fundamental theorem of surface theory** and is expected to play an important part in shell theory.

3_3.4 Third fundamental form

From the last of equations (3_3.51) follows

$$\mathbf{a}_{3,\beta}.\mathbf{a}_{3,\alpha}= b_{\alpha\gamma}\, b^{\gamma}_{\beta}. \qquad (3\_3.56)$$

Therefore

$$d\mathbf{a}_3.d\mathbf{a}_3= b_{\alpha\gamma}\, b^{\gamma}_{\beta}\, d\vartheta^{\alpha} d\vartheta^{\beta} \qquad (3\_3.57)$$

The last result is known as the **third fundamental form** of the surface.

3_3.5 Covariant differentiation : further results.

We add finally some results concerning the covariant differentiation. By analogy to (3_2.88) and (3_2.89), for $\vartheta^3 = 0$ we write

$$a_{\alpha\beta}|_\lambda = a^{\alpha\beta}|_\lambda = \varepsilon_{\alpha\beta}|_\lambda = \varepsilon^{\alpha\beta}|_\lambda = 0. \qquad (3\_3.58)$$

Also by analogy to (3_2.92), the second covariant derivatives of a covariant vectors are

$$v_\alpha|_{\beta\gamma} - v_\alpha|_{\gamma\beta} = R^{\rho}{}_{\cdot\alpha\beta\gamma} v_\rho . \qquad (3\_3.59)$$

In general, the order of covariant derivatives does matter, i.e whenever the order of indices is altered , the final result is altered. Exception to this is when the right hand side of (3_3.59) vanishes, i.e the Riemann_Christoffel tensor vanishes. The latter is possible only when the Gaussian curvature of the surface is zero.

In the special case of the plane the unit vector $\mathbf{a}_3$ is constant and then its derivatives vanish. Thus, the curvature tensors are all zero and consequently the Gaussian curvature is zero.

$\mathbf{a}_3 =$ const. , $\mathbf{a}_{3,\alpha} = 0$

$$b_{\alpha\beta} = 0 \quad , \quad R_{1212} = 0.$$

Using (3_3.43), equation (3_3.59) can be written as

$$v_\alpha|_{\beta\gamma} - v_\alpha|_{\gamma\beta} = a_{\alpha\upsilon}\left[b^\rho_\beta\, b^\upsilon_\gamma - b^\rho_\gamma\, b^\upsilon_\alpha\right] v_\rho \qquad (3\_3.60)$$

which can also be written as

$$\varepsilon^{\alpha\beta} v_\gamma|_{\alpha\beta} = a_{\alpha\upsilon}\varepsilon^{\alpha\rho}\, K\, v_\rho\,. \qquad (3\_3.61)$$

$$\varepsilon^{\alpha\beta} v^\lambda|_{\alpha\beta} = \varepsilon^{\rho\lambda}\, K\, v_\rho$$

Finally, using (3_2.95) the second covariant derivatives of second order surface tensors have the properties

$$T_{\alpha\beta}|_{\rho\eta} - T_{\alpha\beta}|_{\eta\rho} = T_{\lambda\beta}\, R^{\lambda}_{\cdot\alpha\rho\eta} + T_{\alpha\lambda}\, R^{\lambda}_{\cdot\beta\rho\eta}$$

$$T^{\alpha\beta}|_{\rho\eta} - T^{\alpha\beta}|_{\rho\eta} = -\, T^{\lambda\beta}\, R^{\alpha}_{\cdot\lambda\rho\eta} - T^{\alpha\lambda}\, R^{\beta}_{\cdot\lambda\rho\eta} \qquad (3\_3.62)$$

CHAPTER FOUR

STATICS OF SHELLS

4_1 Introduction

The equations of shell theory fall into three categories.

a)_ **Equilibrium**: These equations relate the internal forces and moments referred to the reference surface and the applied loads.

b)_ **Compatibility**: These equations relate the deformation of the reference surface to the displacements of the reference surface.

c)_ **Constitutive relationship**: These equations give the relationship between the internal forces and moments and the deformation of the reference surface.

The equilibrium equations and the compatibility equations are the same for all structures including grid and ribbed shells. The constitutive equations will depend upon the detailed form of the shell and the material properties which may be elastic, elasto_plastic and so on.

In the present chapter, we shall be concerned with the the equilibrium of the structure.

4_2 Equilibrium equations: direct approach

There are basically two ways of deriving the equilibrium equations for shells. One method is to start with the three_dimensional equations of equilibrium and then integrate through the thickness of the shell. The second method, known as the direct approach, deals directly with the equilibrium of the two_dimensional reference surface. All internal forces and moments and applied loads are referred to the reference surface. The direct approach was first used in 1888, when Love formulated his general theory of shells.

According to Naghdi (1963), a modern and fully general derivation of the equations of equilibrium by the direct method was first supplied by Synge and Chien in 1941. Moreover a neat vectorial treatment of Love's derivation was obtained by Reissner also in 1941.

Ericksen and Truesdell (1958) have defined the shell as a simple surface which may be the seat of dynamical actions. The action of one side of any imaginary closed curve on the surface upon the other side is equivalent to field of stress resultant vectors and couple resultant vectors defined on the curve.

The above idea was reproduced in a more explicit form in the treatise by Truesdell and Toupin (1960). They define the shell as a surface or as a region between two surfaces. In both cases the shell is subject to normal forces and tangential forces. As a consequence of this, it will be treated as a surface or a body in three_dimensional space. If the shell is considered as a region enclosed between two surfaces, i.e. with non zero volume, the force and couples on the shell must be derived on the basis of the three_dimensional theory. On the other hand if the shell is considered as a surface of zero volume, one must postulate new forms of the stress principle and the momentum principle.

Before giving a suitable definition of shells which will be adopted in the present work, let us consider the simple case of a curved beam.

In structural mechanics it is conventional to replace the beam by a curved line and to consider the axial force, Shear forces, twisting and bending moments referred to the line, fig.(4_2.1).

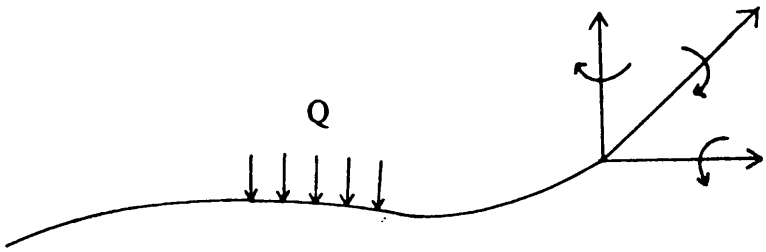
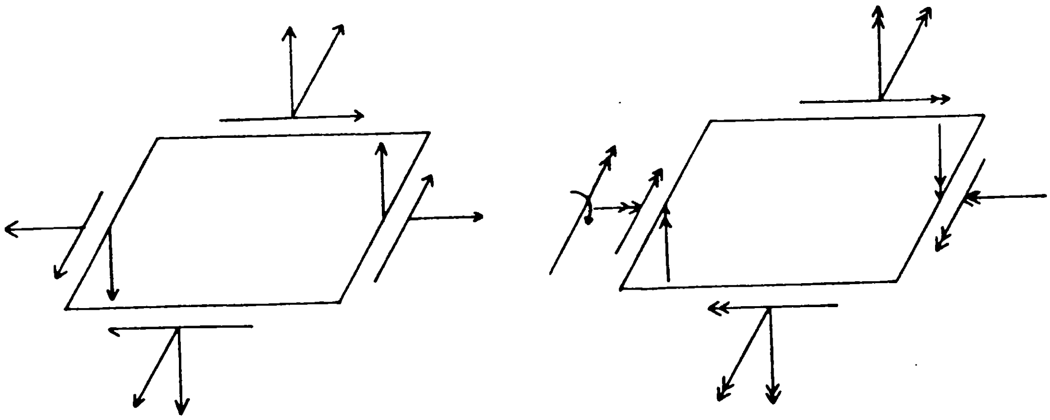


fig.(4_2.1) Beam representation

Then, we assume that there are some relationships between these forces and moments and the deformation of the line. In order for these relations to exist, The cross_sectional dimensions of the beam have to be small compared to the length and the radius of curvature of the beam. However the beam can be of almost any cross_sectional shape and may be made of a combination of materials.

If now, we extend this form of definition to a shell, we replace the real structure by a single surface of no thickness and consider the forces and moments acting upon an element of surface, as is shown in fig.(4_2.2) below.



membrane stresses and shear forces *bending and twisting moments*

fig.(4_2.2)

We can now define a shell as one in which there is a relationship between these forces and moments and the deformation of the reference surface. In some shell structures it is necessary to consider internal degrees of freedom such as the shear deformation of sandwich shells. This relationship will be different for solid shells of constant thickness, grid shells and ribbed shells, and there will always be some degree of approximation.

The main point is that the structure is modelled as a continuous surface so that the individual members of, for instance a grid shell, would be modelled by a two dimensional continuum.

4_3 Geometrical relations

In what follows, the description of the equilibrium of the surface portion of the shell given in the work of Williams (1987) is adopted, except that here the moment about the normal to the surface is considered.

If we imagine a short cut AB made in the surface of the shell, defined by $\mathbf{r}(\vartheta^1,\vartheta^2)$, the direction of which is specified by the vector $d\,\overline{\eta}$. $d\,\overline{\eta}$ lies in the tangent plane of the surface, and perpendicular to AB. The length of the cut is equal to the magnitude of $d\,\overline{\eta}$, fig.(4_3.1).

If we consider now that the portion of surface is under the action of forces and bending moments then these will be represented respectively by the resultants, force **df** and a moment **dm**, which are exerted by one side of the cut on the other. **df** and **dm** depend on the direction and magnitude of $d\,\overline{\eta}$

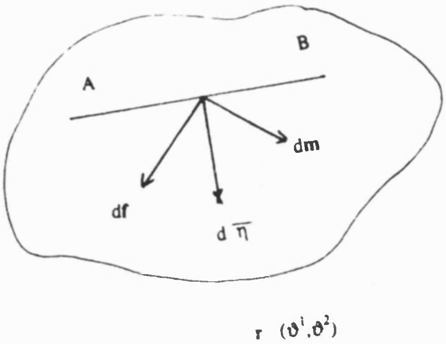


fig.(4_3.1)

$$|d\,\overline{\eta}| = \textit{length of } AB. \tag{4\_3.1}$$

Considering the surface **r** composed from two families of parametric curvilinear coordinates $(\vartheta^1,\vartheta^2)$, hence

$$d\,\overline{\eta} = \mathbf{a}_\beta\; d\eta^\beta = \mathbf{a}^\beta\; d\eta_\beta. \tag{4\_3.2}$$

(It should be remembered that when the indices appears twice in a given expression, it stands for a summation, unless stated otherwise, also the Greek indices range over the values 1 and 2).

$$A\overline{B} = \mathbf{a}_\alpha\; d\vartheta^\alpha = \mathbf{a}^\alpha\; d\vartheta_\alpha \tag{4\_3.3}$$

and also

$$\overline{AB} = \mathbf{n} \times d\,\overline{\eta} \tag{4\_3.4}$$

where $\mathbf{n}(\vartheta^1,\vartheta^2) = \mathbf{a}_3$ represent the vector normal to the surface. Using (4_3.3) and (4_3.4) we get

$$a_\alpha \, d\vartheta^\alpha = \mathbf{n} \times d\,\overline{\eta}. \qquad (4\_3.5)$$

From the base vector product we have

$$\mathbf{n} \times \mathbf{a}_\beta = \varepsilon_{\beta\rho}\, \mathbf{a}^\rho. \qquad (4\_3.6)$$

Using (4_3.5), (4_3.2) and (4_3.6) we get

$$\mathbf{a}_\alpha \, d\vartheta^\alpha = \varepsilon_{\beta\rho}\, a^\rho \, d\eta^\beta. \qquad (4\_3.7)$$

In general, the force resultant vector can be written as

$$df = \text{force} = df^\lambda \, \mathbf{a}_\lambda + df \, \mathbf{n} \qquad (4\_3.8)$$

where df^λ is the tangential component of the force vector and df is its normal component. As, df depends on the magnitude and direction of $d\,\overline{\eta}$, then

$$df^\lambda = n^{\alpha\lambda} \, d\eta_\alpha \qquad (4\_3.9)$$

where $n^{\alpha\lambda}$ is defined as the in_plane stress tensor of second order.

$$df = q^\alpha \, d\eta_\alpha \qquad (4\_3.10)$$

where q^α is the shearing stress tensor. We have also

$$d\,\overline{\eta} = \overline{AB} \times \mathbf{n}. \qquad (4\_3.11)$$

Applying (4_3.3) and (4_3.2) to (4_3.11) then

$$d\,\overline{\eta} = \mathbf{a}_\alpha \, d\vartheta^\alpha \times \mathbf{n} = \varepsilon_{\rho\alpha}\, \mathbf{a}^\rho \, d\vartheta^\alpha = d\eta_\beta \, \mathbf{a}^\beta. \qquad (4\_3.12)$$

Therefore with little manipulation, we end up with

$$d\eta_\beta = \varepsilon_{\beta\alpha}\, d\vartheta^\alpha \quad , \quad d\eta_\alpha = \varepsilon_{\alpha\gamma}\, d\vartheta^\gamma. \tag{4_3.13}$$

Now, a combination of (4_3.9) and (4_3.13) leads to

$$df^\lambda = n^{\alpha\lambda}\, \varepsilon_{\alpha\rho}\, d\vartheta^\rho \tag{4_3.14}$$

and (4_3.10) and (4_3.13) give

$$df = q^\alpha\, \varepsilon_{\alpha\rho}\, d\vartheta^\rho. \tag{4_3.15}$$

Thus, equation (4_3.8) becomes

$$\mathbf{df} = n^{\alpha\lambda}\, \varepsilon_{\alpha\rho}\, d\vartheta^\rho\, .\mathbf{a}_\lambda + q^\alpha\, \varepsilon_{\alpha\rho}\, d\vartheta^\rho\, \mathbf{n} \tag{4_3.16}$$

$$\mathbf{df} = \left(n^{\alpha\lambda}\, \mathbf{a}_\lambda + q^\alpha\, \mathbf{n} \right) \varepsilon_{\alpha\rho}\, d\vartheta^\rho. \tag{4_3.17}$$

The moment **dm** lies in the tangent plane of the surface, hence also depends on $d\overline{\eta}$ and represents the axis about which the moment acts. Similarly to **df**, the vector **dm** will be

$$\mathbf{dm} = dm_\lambda\, \mathbf{a}^\lambda + dm\, \mathbf{n} \tag{4_3.18}$$

where

$$dm_\lambda = \varepsilon_{\lambda\rho} m^{\gamma\rho} d\eta_\gamma \quad , \quad dm = m^{3\gamma} d\eta_\gamma$$

Using (4_3.13), the vector **dm** becomes

$$dm = \varepsilon_{\lambda\rho} m^{\gamma\rho} \varepsilon_{\gamma\alpha}\, d\vartheta^\alpha \mathbf{a}^\lambda + m^{3\gamma} \varepsilon_{\gamma\alpha}\, d\vartheta^\alpha\, \mathbf{n}$$

$$dm = \left[\varepsilon_{\lambda\rho} m^{\gamma\rho} \mathbf{a}^\lambda + m^{3\gamma}\, \mathbf{n} \right] \varepsilon_{\gamma\alpha}\, d\vartheta^\alpha\, . \tag{4_3.19}$$

4_4 Equilibrium of a surface element

Having found the expressions of forces and moments using the geometrical relations, now we proceed to establish the static equilibrium of an infinitesimally small element ABCD, taken from the surface of the shell. Fig.(4_4.1) shows the surface element ABCD subjected to a combination of forces and moments.

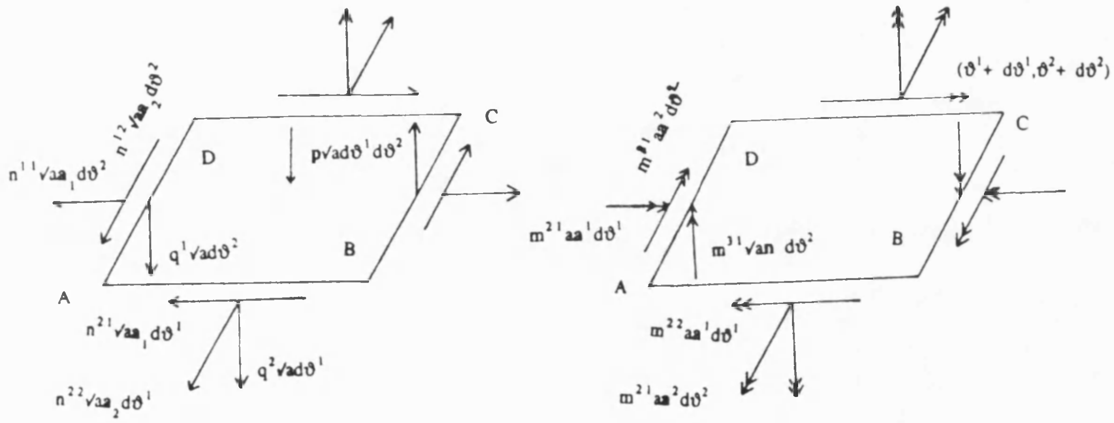


fig.(4_4.1) Equilibrium of the surface element

Let us assume that the shell is subjected to an external load measured per unit area of the middle surface.

$$\mathbf{p} = p^{\alpha}\,\mathbf{a}_{\alpha} + p\,\mathbf{n} \qquad (4\_4.1)$$

where p^α, p are the tangential and normal components to the surface respectively. From equation (3_3.20), the area of the reference surface element ABCD is

$$ds = \sqrt{a}\ d\vartheta^1\ d\vartheta^2 \qquad (4\_4.2)$$

and therefore the external load applied to the element is

$$\left(p^\alpha\ \mathbf{a}_\alpha + p\ \mathbf{n} \right) \sqrt{a}\ \ d\vartheta^1 d\vartheta^2. \qquad (4\_4.3)$$

The equation of equilibrium on the element ABCD is obtained from the summation of the stress components, plus the external load. The force across the side BC is given by (4_3.17) in which ρ is set equal 2

$$\left(n^{1\lambda}\mathbf{a}_\lambda + q^1\mathbf{n} \right) \sqrt{a}\ d\vartheta^2. \qquad (4\_4.4)$$

The force crossing AD is almost exactly equal and opposed to (4_4.4). The sum of the force crossing AD and BC is equal to the rate of change of (4_4.4) with respect to ϑ^1 multiplied by $d\vartheta^1$, which is equal to

$$\frac{\partial}{\partial\ \vartheta^1} \left(\left(n^{1\lambda}\mathbf{a}_\lambda + q^1\mathbf{n} \right) \sqrt{a} \right) d\vartheta^1 d\vartheta^2 \qquad (4\_4.4)_1$$

It is important to realize that in differentiating (4_4.4) with respect to ϑ^1, we automatically take into account the effect of the change in direction of the base vectors and hence includes the effect of curvature of the surface and coordinate curves.

Similarly, the forces across AB and DC by putting $\rho = 1$, are

$$\frac{\partial}{\partial \vartheta^2}\left(\left(n^{2\lambda}\mathbf{a}_\lambda + q^2\mathbf{n}\right) \sqrt{a}\right) d\vartheta^1 d\vartheta^2. \tag{4_4.5}$$

Addition of (4_4.3) to (4_4.4)$_1$ and (4_4.5), gives the totality of forces acting on the element ABCD:

$$\frac{\partial}{\partial \vartheta^\alpha}\left(\left(n^{\alpha\lambda}\mathbf{a}_\lambda + q^\alpha\mathbf{n} \right) \sqrt{a}\right) d\vartheta^1 d\vartheta^2 + \left(p^\alpha\mathbf{a}_\alpha + p\ \mathbf{n} \right)\sqrt{a}\ d\vartheta^1 d\vartheta^2 = 0.$$

Dividing the above equation by $d\vartheta^1 d\vartheta^2$, we obtain

$$\frac{\partial}{\partial \vartheta^\alpha}\left(\left(n^{\alpha\lambda}\mathbf{a}_\lambda + q^\alpha\ \mathbf{n}\right) \sqrt{a}\right) + \left(p^\alpha\mathbf{a}_\alpha + p\ \mathbf{n}\right) \sqrt{a} = 0. \tag{4_4.6}$$

The general equation of equilibrium of forces can be resolved into their components, normal and tangential to the surface, by scalar multiplying it by $\mathbf{n}$ and $\mathbf{a}^\rho$ respectively. Thus, in the normal direction, after simplification and use of relations from (3_3.26) and (3_3.46), we write

$$n^{\alpha\lambda}\ b_{\alpha\lambda} + q^\alpha|_\alpha + p = 0 \tag{4_4.7}$$

where

$$q^\alpha|_\alpha = q^\alpha{}_{,\alpha} + \Gamma^\alpha_{\upsilon\alpha}\ q^\upsilon.$$

In the tangential direction, we get the following two equations for $\rho = 1$ and $\rho = 2$

$$n^{\alpha\rho}|_\alpha - q^\alpha b^\rho_\alpha + p^\rho = 0 \tag{4_4.8}$$

where

$$n^{\alpha\rho}|_{\alpha} = n^{\alpha\rho}{}_{,\alpha} + \Gamma^{\rho}_{\gamma\alpha}\, n^{\gamma\alpha} + \Gamma^{\alpha}_{\gamma\alpha}\, n^{\rho\gamma}.$$

By the same principle as for the forces, the resultant moments across BC and DA (putting $\gamma = 1$ into (4_3.19)) is

$$\frac{\partial}{\partial\vartheta^1}\left[\varepsilon_{\lambda\rho} m^{1\rho}\mathbf{a}^{\lambda} + m^{31}\,\mathbf{n}\right]\sqrt{a}\;\; d\vartheta^2 d\vartheta^1. \qquad (4\_4.9)$$

Similarly, the resultant moment across AB and DC is

$$\frac{\partial}{\partial\vartheta^2}\left[\varepsilon_{\lambda\rho} m^{2\rho}\mathbf{a}^{\lambda} + m^{32}\,\mathbf{n}\right]\sqrt{a}\;\; d\vartheta^1 d\vartheta^2. \qquad (4\_4.10)$$

Adding (4_4.9) and (4_4.10) produces

$$\frac{\partial}{\partial\vartheta^\gamma}\left[\varepsilon_{\lambda\rho} m^{\gamma\rho}\mathbf{a}^{\lambda} + m^{3\gamma}\,\mathbf{n}\right]\sqrt{a}\;\; d\vartheta^1 d\vartheta^2. \qquad (4\_4.11)$$

The shearing and in_plane forces, also produce moments equal to the vector product of forces multiplied by their lever arms, so that across BC we have

$$(\,\mathbf{a}_1 d\vartheta^1)\ \mathrm{x}\ \left[n^{1\alpha}\mathbf{a}_{\alpha} + q^1\mathbf{n}\,\right]\sqrt{a}\, d\vartheta^2 \qquad (4\_4.12)$$

A similar quantity is also obtained in the second direction, so that the total moment due to shearing and in_plane forces is

$$(\,\mathbf{a}_\rho d\vartheta^\rho)\ \mathrm{x}\ \left[n^{\beta\alpha}\mathbf{a}_{\alpha} + q^\beta\mathbf{n}\,\right]\ \varepsilon_{\beta\gamma} d\vartheta^\gamma. \qquad (4\_4.13)$$

(4_4.13), with the use of some relations from (3_3.23), becomes

$$\varepsilon_{\gamma\rho}\left(n^{\gamma\rho}\mathbf{a}_3 - q^{\gamma}\mathbf{a}^{\rho}\right)\sqrt{a}\, d\vartheta^1 d\vartheta^2. \qquad (4\_4.14)$$

It is to be noted that the effect of external loads produce moment terms of higher order, since the vector **p** is first multiplied by the area of the element and then by the lever arm.
Adding (4_4.11) and (4_4.14) together, gives the general equation of moments.

$$\frac{\partial}{\partial\vartheta^{\gamma}}\left[\left[\varepsilon_{\lambda\rho} m^{\gamma\rho}\mathbf{a}^{\lambda} + m^{3\gamma}\,\mathbf{n}\right]\sqrt{a}\right] + \varepsilon_{\gamma\rho}\sqrt{a}\left[n^{\gamma\rho}\mathbf{n} - q^{\gamma}\mathbf{a}^{\rho}\right] = 0. \qquad (4\_4.15)$$

Scalar multiplication of (4_4.15) by **n** and a_{β} also give, one normal and two tangential equations to the surface respectively.
Then, the condition of equilibrium of moments about the surface normal is

$$\varepsilon_{\lambda\rho}\left[m^{\gamma\rho}b^{\lambda}_{\gamma} - n^{\lambda\rho}\right] + m^{3\gamma}|_{\gamma} = 0 \qquad (4\_4.16)$$

where

$$m^{3\gamma}|_{\gamma} = m^{3\gamma}_{,\gamma} + \Gamma^{\gamma}_{\gamma\rho}\, m^{3\rho}.$$

The condition of equilibrium in the tangential directions is

$$\varepsilon_{\beta\rho}\left[m^{\gamma\rho}|_{\gamma} - q^{\rho}\right] - m^{3\gamma}b_{\gamma\beta} = 0 \qquad (4\_4.17)$$

where

$$m^{\gamma\rho}|_{\gamma} = m^{\gamma\rho}_{,\gamma} + \Gamma^{\rho}_{\alpha\gamma}\, m^{\alpha\gamma} + \Gamma^{\gamma}_{\alpha\gamma}\, m^{\rho\alpha}.$$

Thus the set of equilibrium relations can be written as follows

$$\left.\begin{aligned}
&n^{\alpha\lambda}\, b_{\alpha\lambda} + q^{\alpha}|_{\alpha} + p = 0\\
&n^{\alpha\rho}|_{\alpha} - q^{\alpha} b^{\rho}_{\alpha} + p^{\rho} = 0\\
&\varepsilon_{\lambda\rho}\left[m^{\gamma\rho} b^{\lambda}_{\gamma} - n^{\lambda\rho}\right] + m^{3\gamma}|_{\gamma} = 0\\
&\varepsilon_{\beta\rho}\left[m^{\gamma\rho}|_{\gamma} - q^{\rho}\right] - m^{3\gamma} b_{\gamma\beta} = 0
\end{aligned}\right\}. \qquad (4_4.18)$$

These equations are exact in the sense that if compared to those derived on the basis of three_dimensional theory, they have the same structure, Naghdi (1963) and Green & Zerna (1968).

Equations (4_4.18) are the six general equilibrium equations of shell theory. They involve twelve unknowns. Therefore the problem of the theory of thin shells is internally statically indeterminate, i.e. even considering the equilibrium of the infinite element ABCD, the number of equations is insufficient for the determination of the unknowns. Thus, the solution of the problem is impossible, unless a number of the unknowns entering in the equilibrium equations are related through a special law to the deformations.

CHAPTER FIVE

DEFORMATION OF SHELLS

5_1 Introduction

A number of different approaches demonstrating different points of view concerning the deformation and strain measures of surfaces have been suggested for the treatment of the subject of shells and plates. Basically two of them have to be distinguished namely, a derivation based on the general three_dimensional measures of strain and deformation, and a derivation based on the concept of oriented bodies founded by Duhem and adopted later to one and two_dimensional problems by the brothers Cosserat.

In a derivation based on the three_dimensional theory, exact measures of strain are usually derived either using a deformation gradients or using the components of the displacements vector, after the manner of Love. Naghdi (1963) argued that, the strains and deformations derived on the basis of the deformation gradients are not necessarily convenient measures. However, the use of the displacement components enable us in the application of boundary value problems to express the boundary conditions in terms of displacement components.

In the oriented bodies, on the other hand, the basic ingredients for obtaining the Kinematical quantities of the deformation are the vector functions **r** and **d**, which represent respectively the position vector of the surface and the single deformable director. These two vectors are assumed to be differentiable as many times as requested, with respect to t (time) and the surface coordinates ϑ^{α}.

In the present chapter, however, the theory which represents a particular case of the Cosserat model, in which no director is assigned to the material points of the surface, is basically followed. However, the angular velocity of an element of surface is introduced to give kinematical results which will facilitate the discussion of the boundary conditions of shells.

The deformation of the shell is that of the reference surface, just as we assumed before in the statics of shells. Therefore no assumption or approximation is made through the following work, except the shell being two_dimensional.

5_2 Rate of change of surface quantities

A point in the space of fig. (5_2.1) is defined by the following relation

$$\mathbf{R} = \mathbf{r} + \vartheta^3\mathbf{n} \ . \qquad (5\_2.1)$$

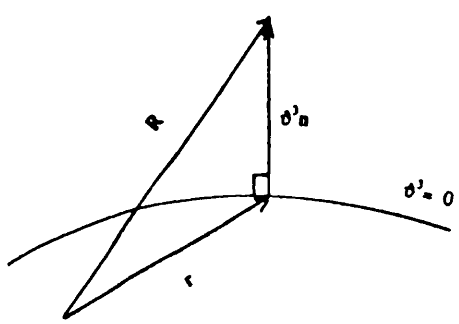


fig.(5_2.1) Position vector of a surface

In (5_2.1) $\mathbf{r}$ and $\mathbf{n}$ depend on the two coordinates $(\vartheta^1,\vartheta^2)$, and the latter is vector of unit magnitude perpendicular to the reference surface.

As we are mainly concerned with a reference surface, then we put $\vartheta^3 = 0$. The surface $\vartheta^3 = 0$ will be defined by the position vector $\mathbf{r}(\vartheta^1,\vartheta^2)$. The position vector $\mathbf{r}(\vartheta^1,\vartheta^2,t)$, will indicate kinematically the position of the deformed surface. The variable t will define the position of the base vectors at any time t, during the process of deformation. It should be noted, however, that the variable t is different from the coordinate ϑ^3, which defines the third dimension of the body.

It is to be noted that the surface geometry given in chapter two, in which the first and second fundamental forms of the surface and some other quantities involving $\mathbf{r}$, $\mathbf{n}$ and their derivatives remain valid, except that now these functions depend on the parameter t characterizing time. In the forthcoming work, a dot over a symbol indicates partial differentiation with respect to time.

Let the vector field v corresponds to the velocity of the surface,and denote

$$\mathbf{v} = v^i \mathbf{a}_i = v_i \mathbf{a}^i$$

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\partial \mathbf{r}}{\partial t} = v^\beta \mathbf{a}_\beta + v\,\mathbf{n} = v_\beta \mathbf{a}^\beta + v\,\mathbf{n} \qquad (5\_2.2)$$

v^β, v_β and v are respectively, the contravariant, covariant and the normal components of the velocity vector v.

The gradients of the velocity vector are its derivatives and are given by

$$v_{,\alpha} = \bar{r}_{,\alpha} = \bar{a}_{\alpha}$$
$$= v^{\beta}_{,\alpha}\, a_{\beta} + v^{\beta}\, a_{\beta,\alpha} + v_{,\alpha}\, n + v\, n_{,\alpha} \qquad (5\_2.3)$$
$$= v_{\beta,\alpha}\, a^{\beta} + v_{\beta}\, a^{\beta}_{,\alpha} + v_{,\alpha}\, n + v\, n_{,\alpha}.$$

Using the formulae of Weingarten and Gauss from (3_3.48), equations (5_2.3) becomes

$$\bar{a}_{\alpha} = v^{\beta}_{,\alpha}\, a_{\beta} + v^{\beta}\, \Gamma^{\lambda}_{\beta\alpha}\, a_{\lambda} + v^{\beta}\, b_{\beta\alpha}\, n + v_{,\alpha}\, n - v\, b^{\lambda}_{\alpha}\, a_{\lambda}$$

$$= v_{\beta,\alpha}\, a^{\beta} - v_{\beta}\, \Gamma^{\beta}_{\alpha\lambda}\, a^{\lambda} + v_{\beta}\, b^{\beta}_{\alpha}\, n + v_{,\alpha}\, n - v\, b_{\alpha\beta}\, a^{\beta}. \qquad (5\_2.4)$$

With little manipulation, we get the rate of change of the base vectors written in the following manner

$$\bar{a}_{\alpha} = \left[v^{\beta}_{,\alpha} + v^{\lambda}\, \Gamma^{\beta}_{\lambda\alpha} - v\, b^{\beta}_{\alpha} \right] a_{\beta} + \left[v_{,\alpha} + v^{\beta}\, b_{\beta\alpha} \right] n$$

$$= \left[v_{\beta,\alpha} - v_{\lambda}\, \Gamma^{\lambda}_{\beta\alpha} - v\, b_{\alpha\beta} \right] a^{\beta} + \left[v_{,\alpha} + v_{\beta}\, b^{\beta}_{\alpha} \right] n. \qquad (5\_2.5)$$

Now, using the concept of covariant differentiation from (3_3.46) we write

$$\dot{\mathbf{a}}_\alpha = \left[v^\beta|_\alpha - v\, b^\beta_\alpha \right] \mathbf{a}_\beta + \left[v|_\alpha + v^\beta\, b_{\beta\alpha} \right] \mathbf{n}$$

$$\dot{\mathbf{a}}_\alpha = \left[v_\beta|_\alpha - v\, b_{\alpha\beta} \right] \mathbf{a}^\beta + \left[v|_\alpha + v_\beta\, b^\beta_\alpha \right] \mathbf{n}. \qquad (5\_2.6)$$

Equations (3_3.7) continue to hold as the surface deforms, then differentiating the first equations of (3_3.7) with respect to time and using (5_2.6) we write

$$\dot{\mathbf{n}} = - \left[v|_\alpha + v_\beta\, b^\beta_\alpha \right] \mathbf{a}^\alpha. \qquad (5\_2.7)$$

Equations (5_2.6) and (5_2.7) are respectively the rate of change of the surface base vectors and the unit normal to the surface.

The rate of change of the metric tensors will be

$$\begin{aligned} \dot{a}_{\alpha\beta} &= \dot{\mathbf{a}}_\alpha.\mathbf{a}_\beta + \mathbf{a}_\alpha.\dot{\mathbf{a}}_\beta \\ &= \mathbf{v},_\alpha.\mathbf{a}_\beta + \mathbf{a}_\alpha.\mathbf{v},_\beta. \end{aligned} \qquad (5\_2.8)$$

Using (5_2.6), equation (5_2.8) becomes

$$\dot{a}_{\alpha\beta} = v_\beta|_\alpha + v_\alpha|_\beta - 2.v\, b_{\alpha\beta}. \qquad (5\_2.9)$$

From the first equation of (3_3.9), the Kronecker delta is constant, then its differentiation with respect to time gives

$$\dot{a}_{\rho\gamma} a^{\gamma\alpha} + a_{\rho\gamma} \dot{a}^{\gamma\alpha} = 0. \qquad (5\_2.10)$$

Equation (5_2.10) with (5_2.9) together give the rate of change of the contravariant metric tensor of the surface in the following form

$$\dot{a}^{\gamma\rho} = - \dot{a}_{\alpha\beta}\, a^{\alpha\gamma}\, a^{\beta\rho}$$

$$\dot{a}^{\gamma\rho} = - \left[v_\beta|_\alpha + v_\alpha|_\beta - 2\, v\, b_{\alpha\beta} \right] a^{\alpha\gamma}\, a^{\beta\rho} \tag{5\_2.11}$$

In chapter three we showed the raising and lowering of indices with respect to tensors, that process will be modified when differentiation with respect to time is considered

$$\dot{A}_\beta^{\cdot\alpha} = \dot{a}^{\alpha\gamma} A_{\beta\gamma} + a^{\alpha\gamma} \dot{A}_{\beta\gamma} \qquad \text{raising}$$

$$\dot{A}_\beta^{\cdot\alpha} = \dot{a}_{\gamma\beta} A^{\gamma\alpha} + a_{\gamma\beta} \dot{A}^{\gamma\alpha}. \qquad \text{lowering} \tag{5\_2.12}$$

The rate of change of the determinant (a) will be

$$\dot{a} = \dot{a}_{11} a_{22} + a_{11} \dot{a}_{22} - 2 a_{12} \dot{a}_{12}. \tag{5\_2.13}$$

Introducing the values of the rate of change of metric tensors from equations (5_2.9) into (5_2.13), we end up with the following expression

$$\dot{a} = 2\left[\left(v_1|_1 - v\, b_{11}\right).a_{22} + \left(v_2|_2 - v\, b_{22}\right).a_{11} - \left(v_2|_1 + v_1|_2 - 2v\, b_{12}\right) a_{12} \right]$$

$$= 2\, a a^{\alpha\beta} \left[v_\alpha|_\beta - v\, b_{\alpha\beta} \right] = 2\, a \left[v^\alpha|_\alpha - v\, b_\alpha^\alpha \right] \tag{5\_2.14}$$

For the rate of change of the reciprocal base vectors, then from the first equation of (3_3.8) we have

$$\dot{\mathbf{a}}^\gamma = \dot{\overline{a^{\gamma\rho} \mathbf{a}_\rho}} = \dot{a}^{\gamma\rho} \mathbf{a}_\rho + a^{\gamma\rho} \dot{\mathbf{a}}_\rho.$$

Using the first equation of (5_2.11), we write finally

$$\bar{a}^{\gamma} = a^{\gamma\lambda} \left\{ \left[v|_{\lambda} + v_{\beta} b^{\beta}_{\lambda} \right] \mathbf{n} - \left[v_{\lambda}|_{\beta} - v\, b_{\beta\lambda} \right] \mathbf{a}^{\beta} \right\}. \tag{5_2.15}$$

Lastly, the rate of change of an element of area dS is obtained from the differentiation with respect to time of (3_3.20) and use of (5_2.14)

$$dS^{\cdot} = \frac{1}{2\sqrt{a}} \bar{a}\, d\vartheta^1 d\vartheta^2$$

$$= \sqrt{a} \left[v^{\alpha}|_{\alpha} - v\, b^{\alpha}_{\alpha} \right] dS \tag{5_2.16}$$

5_3 Rate of membrane strain tensor

The surface $\mathbf{r}$ when subject to deformation, may undergo elongation or contraction in its plane. The line element of the surface is given in (3_3.15), and is

$$\delta s^2 = \delta \mathbf{r}.\delta \mathbf{r} = a_{\alpha\beta}\, \delta\vartheta^{\alpha}\delta\vartheta^{\beta}.$$

Differentiation of both sides with respect to time gives

$$2\, \delta s \delta \bar{s} = \bar{a}_{\alpha\beta}\, \delta\vartheta^{\alpha}\delta\vartheta^{\beta}. \tag{5_3.1}$$

Using (5_2.9), we write

$$\delta s\, \delta \bar{s} = \left[\frac{v_{\beta}|_{\alpha} + v_{\alpha}|_{\beta}}{2} - v\, b_{\alpha\beta} \right] \delta\vartheta^{\alpha}\delta\vartheta^{\beta}$$

$$= Y_{\alpha\beta}\, \delta\vartheta^{\alpha}\delta\vartheta^{\beta} \tag{5_3.2}$$

where

$$\gamma_{\alpha\beta} = \gamma_{\beta\alpha} = \frac{\dot{a}_{\alpha\beta}}{2} = \left[\frac{v_{\beta}|_{\alpha} + v_{\alpha}|_{\beta}}{2} - v\, b_{\alpha\beta}\right]. \qquad (5\_3.3)$$

Equation (5_3.3) is the rate of the membrane strain tensor, which has three independent components, γ_{11}, $\gamma_{12} = \gamma_{21}$ and γ_{22}.

Dividing the second equation of (5_3.2) by δs^2

$$\frac{\dot{\delta s}}{\delta s} = \frac{\gamma_{\alpha\beta}\ \delta\vartheta^{\alpha}\delta\vartheta^{\beta}}{a_{\lambda\rho}\ \delta\vartheta^{\lambda}\delta\vartheta^{\rho}}. \qquad (5\_3.4)$$

Let us imagine three adjacent points A, B and C which lie on and move with the surface. We will further imagine that the line CA is instantaneously perpendicular to AB and has the same length as AB at the same time at which we examine the surface. We will now find the rate of change of the angle, α, between AB and AC which is equal to $\pi/2$ at the instant we are considering, fig.(5_3.1).

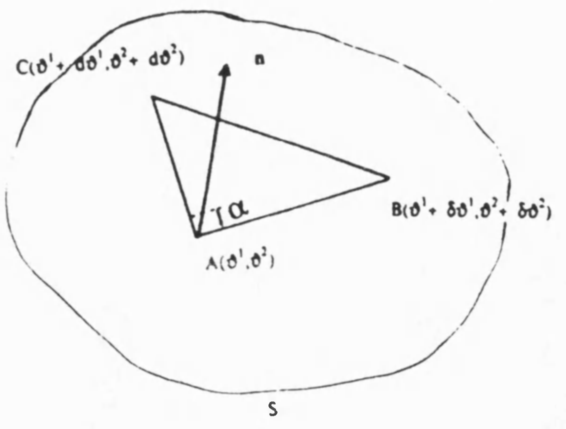


fig.(5_3.1)

From (3_3.14) and the vector product rule

$$AC = \mathbf{n} \times AB$$

$$a_\beta d\vartheta^\beta = \mathbf{n} \times a_\rho \delta\vartheta^\rho$$

$$d\vartheta^\beta = a^{\lambda\beta}\, \varepsilon_{\rho\lambda}\, \delta\vartheta^\rho \qquad (5\_3.5)$$

and the angle, α, between these two lines is given from (3_3.18) as

$$\cos\alpha = \frac{\left[a_\alpha\ \delta\vartheta^\alpha\right] \cdot \left[a_\beta d\vartheta^\beta\right]}{\sqrt{\left[a_{\lambda\upsilon}\ \delta\vartheta^\lambda \delta\vartheta^\upsilon\right]}\ \sqrt{\left[a_{\eta\psi}\ d\vartheta^\eta d\vartheta^\psi\right]}}$$

$$= \frac{a_{\alpha\beta}\delta\vartheta^\alpha d\vartheta^\beta}{\sqrt{\left[a_{\lambda\upsilon}\ \delta\vartheta^\lambda \delta\vartheta^\upsilon\right]}\ \sqrt{\left[a_{\eta\psi}\ d\vartheta^\eta d\vartheta^\psi\right]}}. \qquad (5\_3.6)$$

In differentiating (5_3.6) with respect to time $\delta\vartheta^\alpha$ and $d\vartheta^\beta$ are taken as constants since the points are convected, that is move with the surface. Thus since $a_{\alpha\beta}\delta\vartheta^\alpha d\vartheta^\beta$ is instantaneously equal to zero,

$$-\left(\frac{\dot{\alpha}}{2}\right) = \frac{\dot{a}_{\alpha\beta}\delta\vartheta^\alpha d\vartheta^\beta}{2\sqrt{\left[a_{\lambda\upsilon}\ \delta\vartheta^\lambda \delta\vartheta^\upsilon\right]}\ \sqrt{\left[a_{\eta\psi} d\vartheta^\eta d\vartheta^\psi\right]}}. \qquad (5\_3.7)$$

Using (5_3.3) and (5_3.5), equation (5_3.7) becomes

$$-\frac{\dot{\alpha}}{2} = \frac{Y_{\alpha\beta} a^{\lambda\beta}\ \varepsilon_{\rho\lambda}\ \delta\vartheta^\rho \delta\vartheta^\alpha}{a_{\lambda\upsilon}\ \delta\vartheta^\lambda \delta\vartheta^\upsilon}. \qquad (5\_3.8)$$

Equation (5_3.4) and (5_3.8) have the same structure as the equations given in (3_3.31) which represent the normal and twist curvatures. They are the rates of direct strain and shear strains respectively, they both depend on the tensor $Y_{\alpha\beta}$ in the same way as the curvature and twist depend on the second order tensor $b_{\alpha\beta}$.

5_4 Membrane strain tensor

By analogy to the rate of the membrane strain tensor, which is found to be one half the rate of change of the metric tensor, the membrane strain tensor will be given by

$$G_{\alpha\beta} = \frac{1}{2}\left[a_{\alpha\beta} - A_{\alpha\beta}\right] \qquad (5\_4.1)$$

where $a_{\alpha\beta}$ is the deformed metric tensor (final state of the deformed surface at some fixed time) which is function of ϑ^{α} and t. Whereas $A_{\alpha\beta}$ is the value of the metric tensor in some reference configuration (undeformed metric tensor) which is independent of t. Also, it is to be noted that

$$\dot{G}_{\alpha\beta} = Y_{\alpha\beta} = \frac{\dot{a}_{\alpha\beta}}{2} \qquad (5\_4.2)$$

5_5 The concept of angular velocities

The deformation of surfaces induces not only stretches but also rotations, and as we used velocities instead of simple displacement, let us introduce the concept of angular velocities.
From (5_3.3) we have

$$vb_{\alpha\beta} = -Y_{\alpha\beta} + \frac{v_{\beta}|_{\alpha} + v_{\alpha}|_{\beta}}{2} \qquad (5\_5.1)$$

substituting the above quantity in the second equation of (5_2.6) we get the following expression for the derivative of the velocity

$$\mathbf{v}_{,\alpha} = Y_{\alpha\beta}\, \mathbf{a}^{\beta} + \left[\frac{v_{\beta}|_{\alpha} - v_{\alpha}|_{\beta}}{2} \right] \mathbf{a}^{\beta} + \left[v|_{\alpha} + v_{\beta}\, b^{\beta}_{\alpha} \right] \mathbf{n} \,. \qquad (5\_5.2)$$

We next introduce a new scalar quantity based on the derivatives of the velocity i.e the velocity gradient $\mathbf{v}_{,\alpha}$

$$\Omega = - \frac{\varepsilon^{\rho\alpha} \mathbf{a}_{\rho} . \mathbf{v}_{,\alpha}}{2} \,. \qquad (5\_5.2)_1$$

Substituting the value of $\mathbf{v}_{,\alpha}$ from (5_2.6) into the above expression we get

$$\Omega = - \frac{\varepsilon^{\rho\alpha} \mathbf{a}_{\rho} . \left[\left[v_{\beta}|_{\alpha} - v\, b_{\alpha\beta} \right] \mathbf{a}^{\beta} + \left[v|_{\alpha} + v_{\beta}\, b^{\beta}_{\alpha} \right] \mathbf{n} \right]}{2} \qquad (5\_5.3)$$

with $b_{\alpha\beta} = b_{\beta\alpha}$, then (5_5.3) becomes

$$\Omega = - \frac{\varepsilon^{\rho\alpha}\, v_{\rho}|_{\alpha}}{2} \,. \qquad (5\_5.4)$$

Also we introduce a second pair of quantities, Ω^{β} defined as

$$\Omega^{\beta} = - \varepsilon^{\alpha\beta} \mathbf{n} . \mathbf{v}_{,\alpha} . \qquad (5\_5.5)$$

Again substituting the value of $\mathbf{v}_{,\alpha}$ from (5_2.6) into (5_5.5), we write

$$\Omega^{\beta} = - \varepsilon^{\alpha\beta}\left[v|_{\alpha} + v_{\rho}\, b^{\rho}_{\alpha} \right]. \qquad (5\_5.6)$$

Equations (5_5.4) and (5_5.6) will be written as follows:

$$\Omega\, \varepsilon_{\alpha\beta} = \left[\frac{v_{\beta}|_{\alpha} - v_{\alpha}|_{\beta}}{2} \right] \qquad (5\_5.7)$$

$\Omega\, \varepsilon_{\alpha\beta} =$

$$\Omega^{\beta}\varepsilon_{\alpha\beta} = - \left[v_{\beta}\, b^{\beta}_{\alpha} + v|_{\alpha} \right]. \qquad (5\_5.8)$$

Now, substitutions of (5_5.7) and (5_5.8) into (5_5.2), gives

$$v_{,\alpha} = Y_{\alpha\beta}\, a^{\beta} + \Omega\, \varepsilon_{\alpha\beta} a^{\beta} - \Omega^{\beta}\varepsilon_{\alpha\beta}\, \mathbf{n}\ . \qquad (5\_5.9)$$

From equations (3_3.23), we substitute the permutation symbols by their values as base vector products, then equation (5_5.9) becomes

$$v_{,\alpha} = Y_{\alpha\beta}\, a^{\beta} + \Omega\, (\mathbf{n} \times \mathbf{a}_{\alpha}) - \Omega^{\beta}(\mathbf{a}_{\alpha} \times \mathbf{a}_{\beta})$$

$$= Y_{\alpha\beta}\, a^{\beta} + \left[\Omega^{\beta}\mathbf{a}_{\beta} + \Omega\, \mathbf{n} \right] \times \mathbf{a}_{\alpha}. \qquad (5\_5.10)$$

The quantity in bracket represents a space vector $\overline{\Omega}$, i.e

$$\overline{\Omega} = \Omega^{\beta}\mathbf{a}_{\beta} + \Omega\, \mathbf{n}. \qquad (5\_5.11)$$

Differentiation of (5_5.11) with respect to α gives

$$\Omega_{,\alpha} = \Omega^{\beta}{}_{,\alpha}\, a_{\beta} + \Omega^{\beta}\, a_{\beta,\alpha} + \Omega_{,\alpha}\, n + \Omega\, n_{,\alpha}. \qquad (5\_5.12)$$

From (3_3.48), we make use of the two formulae of Gauss and Weingarten. We note then, the two following special results for future convenience

$$\Omega_{,\alpha} \cdot a^{\beta} = \Omega^{\beta}|_{\alpha} - \Omega\, b^{\beta}_{\alpha} \qquad (5\_5.13)$$

$$\Omega_{,\alpha} \cdot n = \Omega^{\beta} b_{\beta\alpha} + \Omega|_{\alpha}. \qquad (5\_5.14)$$

Equations (5_5.10) and (5_5.11) together form

$$v_{,\alpha} = Y_{\alpha\beta}\, a^{\beta} + \Omega \times a_{\alpha} \qquad (5\_5.15)$$

where the right hand side of the velocity gradient is composed from two parts. $Y_{\alpha\beta}\, a^{\beta}$ represents the rate of membrane strain and $\Omega \times a_{\alpha}$, due to the vector Ω which represents an angular velocity of the surface, fig.(5_5.1)

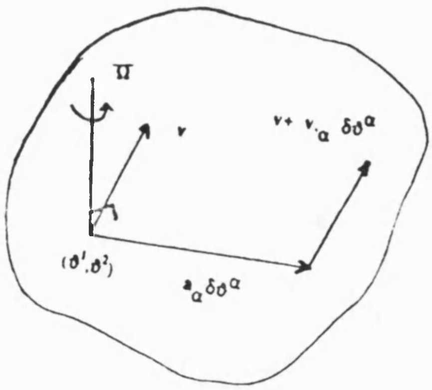


fig.(5_5.1)

The meaning of equation (5_5.15) and particularly the term of the angular velocity can be better explained by the following arguments. Let us imagine the location of two adjacent points A and B on the surface with coordinates ϑ^{α} and $\vartheta^{\alpha}+ \delta\vartheta^{\alpha}$. The line AB between these points is perpendicular to the surface normal **n** and as the surface deforms both lines AB and the normal rotate but remain perpendicular to each other. The rate of change of the unit normal was expressed in (5_2.7) on the basis of

$$\left[\mathbf{n} \, . \, \mathbf{a}_{\alpha} \right]^{\bullet} = 0,$$

then with the use of (5_5.8), it becomes

$$\dot{\mathbf{n}} \;=\; \Omega^{\beta}\varepsilon_{\alpha\beta}\mathbf{a}^{\alpha}. \tag{5\_5.16}$$

As the normal retains its original length after deformation, then $\dot{\mathbf{n}}$ must lie in the plane of the surface. Then the component of the angular velocity of the pair of lines, AB and **n**, in the plane of the surface is given by

$$\mathbf{n} \times \dot{\mathbf{n}} = \Omega^{\beta}\varepsilon_{\alpha\beta}\mathbf{n} \times \mathbf{a}^{\alpha} = \Omega^{\beta}\mathbf{a}_{\beta}.$$

The normal component of the angular velocity of the pair of lines, AB and **n**, i.e. their rotation about the normal, is given by

$$\frac{\left[(\mathbf{a}_{\alpha}\delta\vartheta^{\alpha}) \times (\mathbf{v},_{\beta}\delta\vartheta^{\beta}) \right] .\mathbf{n}}{a_{\lambda\gamma}\delta\vartheta^{\lambda}\delta\vartheta^{\gamma}} = \frac{(\mathbf{a}_{\alpha} \times \mathbf{v},_{\beta}) \, . \, \mathbf{n} \;\delta\vartheta^{\alpha}\,\delta\vartheta^{\beta}}{a_{\lambda\gamma}\delta\vartheta^{\lambda}\delta\vartheta^{\gamma}} \, .$$

Using (3_3.23), (5_5.2) and (5_5.7), we write the above expression after having simplified it as

$$= \Omega + \frac{\varepsilon_{\alpha\mu}\ a^{\mu\rho}\ Y_{\beta\rho}\ \delta\vartheta^{\alpha}\ \delta\vartheta^{\beta}}{a_{\lambda\gamma}\delta\vartheta^{\lambda}\delta\vartheta^{\gamma}} \qquad (5\_5.16)_1$$

which is the normal component of angular velocity plus the rate of shear strain given in equation (5_3.8).

Differentiation of (5_5.15) with respect to λ and then use of the principle of covariant differentiation, gives

$$v_{,\alpha\lambda} = Y_{\alpha\beta}|_{\lambda}\ a^{\beta} + Y_{\alpha\beta}\ b^{\beta}_{\lambda}\ \mathbf{n} + \boldsymbol{\Omega}_{,\lambda} \times \mathbf{a}_{\alpha} + \boldsymbol{\Omega} \times \mathbf{a}_{\alpha,\lambda}. \qquad (5\_5.17)$$

Now interchanging α and λ in the above equation and subtracting, we write.

$$\varepsilon^{\alpha\lambda}\left[Y_{\alpha\beta}|_{\lambda}\ a^{\beta} + Y_{\alpha\beta}\ b^{\beta}_{\lambda}\ \mathbf{n} + \boldsymbol{\Omega}_{,\lambda} \times \mathbf{a}_{\alpha}\right] = 0. \qquad (5\_5.18)$$

By replacing the value of $\boldsymbol{\Omega}$ from (5_5.11) into (5_5.18), we obtain

$$\varepsilon^{\alpha\lambda}\left[Y_{\alpha\beta}|_{\lambda}\ a^{\beta} + Y_{\alpha\beta}\ b^{\beta}_{\lambda}\ \mathbf{n} + \left(\Omega^{\beta}\mathbf{a}_{\beta} + \Omega\ \mathbf{n}\right)_{,\lambda} \times \mathbf{a}_{\alpha}\right] = 0$$

$$\varepsilon^{\alpha\lambda}\left[Y_{\alpha\beta}|_{\lambda}a^{\beta} + Y_{\alpha\beta}\ b^{\beta}_{\lambda}\mathbf{n} + \left(\Omega^{\beta}_{,\lambda}\mathbf{a}_{\beta} + \Omega^{\beta}\mathbf{a}_{\beta,\lambda} + \Omega_{,\lambda}\mathbf{n} + \Omega\ \mathbf{n}_{,\lambda}\right) \times \mathbf{a}_{\alpha}\right] = 0. \qquad (4\_5.19)$$

Again, by using the formulae of Gauss and Weingarten, we write

$$\varepsilon^{\alpha\lambda}\Big[Y_{\alpha\beta}|_{\lambda}a^{\beta} + Y_{\alpha\beta}\ b^{\beta}_{\lambda}\mathbf{n} +$$

$$+\left(\Omega^{\beta}|_{\lambda}\mathbf{a}_{\beta} + \Omega^{\beta}\ b_{\beta\lambda}\mathbf{n} + \Omega|_{\lambda}\mathbf{n} - \Omega\ b^{\beta}_{\lambda}\ \mathbf{a}_{\beta}\right) \times \mathbf{a}_{\alpha}\Big] = 0$$

$$\varepsilon^{\alpha\lambda}\Big\{Y_{\alpha\gamma}|_{\lambda}\mathbf{a}^{\gamma}+\left[\Omega^{\beta}\, b_{\beta\lambda}+\Omega|_{\lambda}\right]\varepsilon_{\alpha\gamma}\mathbf{a}^{\gamma}+$$

$$+\left[\Omega^{\beta}|_{\lambda}-\Omega\, b^{\beta}_{\lambda}\right]\varepsilon_{\beta\alpha}\mathbf{n}+Y_{\alpha\beta}\, b^{\beta}_{\lambda}\,\mathbf{n}\Big\}=0 \qquad (5\_5.20)$$

where

$$\Omega^{\beta}|_{\lambda}=\Omega^{\beta}{}_{,\lambda}+\Omega^{\rho}\,\Gamma^{\beta}_{\rho\lambda}.$$

Scalar multiplication of (5_5.20) by $\mathbf{a}_{\gamma}$ and $\mathbf{n}$ respectively gives

$$\varepsilon^{\alpha\lambda}Y_{\alpha\gamma}|_{\lambda}+\Omega^{\beta}\, b_{\beta\gamma}+\Omega|_{\gamma}=0 \qquad (5\_5.21)$$

$$\left[\Omega^{\lambda}|_{\lambda}-\Omega\, b^{\lambda}_{\lambda}\right]+\varepsilon^{\alpha\lambda}\,Y_{\alpha\beta}\, b^{\beta}_{\lambda}=0. \qquad (5\_5.22)$$

The above two equations (5_5.21), (5_5.22), become when comparing them to (5_5.13) and (5_5.14)

$$\boldsymbol{\Omega}_{,\gamma}\mathbf{n}=-\varepsilon^{\alpha\lambda}Y_{\alpha\gamma}|_{\lambda} \qquad (5\_5.23)$$

$$\boldsymbol{\Omega}_{,\gamma}\mathbf{a}^{\gamma}=\varepsilon^{\alpha\lambda}\,Y_{\alpha\beta}\, b^{\beta}_{\lambda}. \qquad (5\_5.24)$$

5_6 The rate of bending tensor

Consider the rate of bending tensor, expressed as the following second order surface tensor

$$B^{\alpha\beta}=\varepsilon^{\alpha\gamma}\,\boldsymbol{\Omega}_{,\gamma}\mathbf{a}^{\beta}. \qquad (5\_6.1)$$

Using (5_5.13), equation (5_6.1) becomes

$$\beta^{\alpha\beta} = \varepsilon^{\alpha\gamma}\left[\ \Omega^{\beta}|_{\gamma} - \Omega b^{\beta}_{\gamma}\ \right]. \qquad (5\_6.2)$$

Now if we multiply, (5_6.1) by $\varepsilon_{\alpha\beta}$ then, use (5_5.24), we take

$$\varepsilon_{\alpha\beta}\ \beta^{\alpha\beta} = \varepsilon_{\alpha\beta}\varepsilon^{\alpha\gamma}\,\overline{\Omega},_{\gamma}a^{\beta} = \overline{\Omega},_{\beta}.a^{\beta} \qquad (5\_6.3)$$

$$\varepsilon_{\alpha\beta}\ \beta^{\alpha\beta} = \varepsilon^{\alpha\lambda}\ Y_{\alpha\beta}\ b^{\beta}_{\lambda} \qquad (5\_6.4)$$

Therefore, $\quad \beta^{\alpha\beta} \neq \beta^{\beta\alpha}$

we proceed to evaluate the rate of change of the coefficients of the second fundamental form of the surface, we have from (3_3.26) and (3_3.27)

$$b_{\alpha\lambda} = \mathbf{a}_{\alpha,\lambda}.\mathbf{n} \qquad , \qquad b^{\alpha}_{\lambda} = a^{\alpha\beta}b_{\lambda\beta}.$$

Then, the rate of change of $b_{\alpha\lambda}$ is

$$\bar{b}_{\alpha\lambda} = \bar{\mathbf{a}}_{\alpha,\lambda}.\mathbf{n} + \mathbf{a}_{\alpha,\lambda}.\bar{\mathbf{n}}$$

$$\bar{b}_{\alpha\lambda} = \mathbf{v},_{\alpha\lambda}.\mathbf{n} + \mathbf{a}_{\alpha,\lambda}.\bar{\mathbf{n}}. \qquad (5\_6.5)$$

After substituting (5_5.16) and (5_5.17) into equation (5_6.5), we write

$$\bar{b}_{\alpha\lambda} = Y_{\alpha\beta}\ b^{\beta}_{\lambda} + \left[\overline{\Omega},_{\lambda} \times \mathbf{a}_{\alpha} + \overline{\Omega} \times \mathbf{a}_{\alpha,\lambda}\right].\mathbf{n} + \Omega^{\beta}\varepsilon_{\rho\beta}\Gamma^{\rho}_{\alpha\lambda}$$

$$= Y_{\alpha\beta}\ b^{\beta}_{\lambda} + \overline{\Omega},_{\lambda}.\left[\mathbf{a}_{\alpha} \times \mathbf{n}\right] + \overline{\Omega}\left\{\left[\Gamma^{\rho}_{\alpha\lambda}\ \mathbf{a}_{\rho} + b_{\alpha\lambda}\mathbf{n}\ \right] \times \mathbf{n}\right\} + \Omega^{\beta}\varepsilon_{\rho\beta}\Gamma^{\rho}_{\alpha\lambda}$$

$$= Y_{\alpha\beta}\ b^{\beta}_{\lambda} + \overline{\Omega},_{\lambda}.\varepsilon_{\rho\alpha}\mathbf{a}^{\rho} + \overline{\Omega}\,\Gamma^{\rho}_{\alpha\lambda}\varepsilon_{\nu\rho}\mathbf{a}^{\nu} + \Omega^{\beta}\varepsilon_{\rho\beta}\Gamma^{\rho}_{\alpha\lambda}$$

$$\dot{b}_{\alpha\lambda} = Y_{\alpha\beta}\, b^{\beta}_{\lambda} + \Omega_{,\lambda}\, \varepsilon_{\rho\alpha} a^{\rho} + \Omega^{\beta}\, \Gamma^{\rho}_{\alpha\lambda}\, \varepsilon_{\beta\rho} + \Omega^{\beta} \varepsilon_{\rho\beta} \Gamma^{\rho}_{\alpha\lambda}.$$

Then, using (5_6.1), we end up with

$$\dot{b}_{\alpha\lambda} = Y_{\alpha\beta}\, b^{\beta}_{\lambda} + \varepsilon_{\rho\alpha}\, \varepsilon_{\beta\lambda}\, B^{\beta\rho}. \tag{5\_6.6}$$

To find the rate of change of the tensor b^{α}_{λ}, we also write

$$\dot{b}^{\alpha}_{\lambda} = \dot{a}^{\alpha\beta} b_{\lambda\beta} + a^{\alpha\beta} \dot{b}_{\lambda\beta}. \tag{5\_6.7}$$

Substituting (5_2.11) and (5_6.6) into (5_6.7), then

$$\dot{b}^{\alpha}_{\lambda} = -\,2\, Y_{\gamma\rho}\, a^{\gamma\alpha}\, a^{\rho\beta} b_{\lambda\beta} + \left[Y_{\lambda\gamma}\, b^{\gamma}_{\beta} + \varepsilon_{\rho\lambda}\, \varepsilon_{\gamma\beta}\, B^{\gamma\rho} \right] a^{\alpha\beta}$$

$$= -\, Y_{\gamma\lambda} b^{\gamma\alpha} + \varepsilon_{\rho\lambda}\, \varepsilon_{\gamma\beta}\, B^{\gamma\rho}\, a^{\alpha\beta}. \tag{5\_6.8}$$

Differentiating (3_3.37) with respect to time, we write the rate of change of the mean curvature in the following form

$$\dot{H} = \frac{\dot{b}^{\alpha}_{\alpha}}{2} = \frac{-\, Y_{\rho\alpha} b^{\rho\alpha} + \varepsilon_{\rho\alpha}\, \varepsilon_{\gamma\beta}\, B^{\gamma\rho}\, a^{\alpha\beta}}{2}. \tag{5\_6.9}$$

From (3_3.38) the rate of change of the Gaussian curvature is

$$\dot{K} = \dot{b}^{1}_{1}\, b^{2}_{2} + b^{1}_{1}\, \dot{b}^{2}_{2} - \dot{b}^{1}_{2}\, b^{2}_{1} - b^{1}_{2}\, \dot{b}^{2}_{1} = \dot{b}^{\alpha}_{\alpha}\, b^{\beta}_{\beta} - \dot{b}^{\beta}_{\alpha}\, b^{\alpha}_{\beta}$$

using (5_6.8), we write

$$\bar{K} = \left[- Y_{\alpha\lambda} b^{\lambda\alpha} + \varepsilon_{\rho\alpha}\, \varepsilon_{\gamma\upsilon}\, \beta^{\gamma\rho}\, a^{\alpha\upsilon}\right] b^{\beta}_{\beta} - \left[- Y_{\alpha\lambda} b^{\lambda\beta} + \varepsilon_{\rho\alpha}\, \varepsilon_{\gamma\upsilon}\, \beta^{\gamma\rho}\, a^{\beta\upsilon}\right] b^{\alpha}_{\beta}$$

$$= Y_{\alpha\lambda} \left[-b^{\lambda\alpha}\, b^{\beta}_{\beta} + b^{\lambda\beta} b^{\alpha}_{\beta}\right] + \varepsilon_{\rho\alpha}\, \varepsilon_{\gamma\upsilon}\, \beta^{\gamma\rho}\left[a^{\alpha\upsilon} b^{\beta}_{\beta} - a^{\beta\upsilon} b^{\alpha}_{\beta}\right]$$

also using (3_3.22)

$$= Y_{\alpha\lambda} \left[-b^{\lambda\alpha}\, b^{\beta}_{\beta} + b^{\lambda\beta} b^{\alpha}_{\beta}\right] + \varepsilon_{\rho\alpha}\varepsilon^{\xi\zeta}\, a_{\gamma\xi}\, a_{\upsilon\zeta}\, \beta^{\gamma\rho}\left[a^{\alpha\upsilon} b^{\beta}_{\beta} - a^{\beta\upsilon} b^{\alpha}_{\beta}\right]$$

$$= Y_{\alpha\lambda}\left[-b^{\lambda\alpha}\, b^{\beta}_{\beta} + b^{\lambda\beta} b^{\alpha}_{\beta}\right] + \left[a_{\gamma\rho} a_{\upsilon\alpha} - a_{\gamma\alpha}\, a_{\upsilon\rho}\right] \beta^{\gamma\rho}\left[a^{\alpha\upsilon} b^{\beta}_{\beta} - a^{\beta\upsilon} b^{\alpha}_{\beta}\right]$$

with the following in mind

$$a_{\gamma\rho} a_{\upsilon\alpha} a^{\alpha\upsilon} b^{\beta}_{\beta} = a_{\gamma\rho} \delta^{\alpha\upsilon}_{\alpha\upsilon}\, b^{\beta}_{\beta} = 2a_{\gamma\rho} b^{\beta}_{\beta} \quad , \quad a_{\gamma\rho} a_{\upsilon\alpha} a^{\beta\upsilon} b^{\alpha}_{\beta} = a_{\gamma\rho} \delta^{\beta}_{\alpha} b^{\alpha}_{\beta}$$

$$a_{\gamma\alpha}\, a_{\upsilon\rho} a^{\alpha\upsilon} b^{\beta}_{\beta} = a_{\gamma\alpha}\, \delta^{\alpha}_{\rho}\, b^{\beta}_{\beta} \qquad , \quad a_{\gamma\alpha} a_{\upsilon\rho} a^{\beta\upsilon} b^{\alpha}_{\beta} = a_{\gamma\alpha} \delta^{\beta}_{\rho}\, b^{\alpha}_{\beta}.$$

Then,

$$\bar{K} = Y_{\alpha\lambda}\left[-b^{\lambda\alpha}\, b^{\beta}_{\beta} + b^{\lambda\beta} b^{\alpha}_{\beta}\right] + \left\{\left[2\, a_{\gamma\rho} - \delta^{\alpha}_{\rho} a_{\gamma\alpha}\right] b^{\beta}_{\beta} - \left[\delta^{\beta}_{\alpha}\, a_{\gamma\rho} - \delta^{\beta}_{\rho}\, a_{\gamma\alpha}\right] b^{\alpha}_{\beta}\right\} \beta^{\gamma\rho}$$

and finally, we have

$$\bar{K} \quad = Y_{\beta\lambda}\left[b^{\lambda\rho} b^{\beta}_{\rho} - b^{\lambda\beta} b^{\rho}_{\rho}\right] + b_{\lambda\rho}\beta^{\lambda\rho}. \qquad (5\_6.10)$$

5_6.1 The compatibility equations

We defined before the compatibility equations as the relations between the deformation of the reference surface and the overall displacements. The deformation of the reference surface has been expressed by the bending tensor $\beta^{\lambda\rho}$ and the membrane strain tensor $\gamma_{\beta\lambda}$. These tensors are also functions of displacements and rotations of the surface which involves the first and second fundamental forms of the surface that are related by Gauss and Codazzi equations. Thus we also expect to find a relations between the membrane and bending tensors to ensure the continuity of deformation of the surface.

As we mentioned before, the Gaussian curvature is a bending invariant, therefore any change in its final expression is due to the change of lengths and angles corresponding to the intrinsic geometry of the surface. Gauss's theorem permits writing the expression of Gaussian curvature in terms of the coefficients of the first fundamental form only. Thus, from equation (5_6.10) we expect to be able to find a relation between the rate of bending tensor and the rate of membrane strain tensor.

Then, substituting the value of $\beta^{\alpha\beta}$ from (5_6.2) into the second term of equation (5_6.10), we write

$$b_{\lambda\rho}\beta^{\lambda\rho} = b_{\lambda\rho}\varepsilon^{\lambda\gamma}\Big[\,\Omega^{\rho}|_{\gamma} - \Omega b^{\rho}_{\gamma}\,\Big].$$

The values of the terms in bracket are substituted from (5_5.6) and (5_5.4), hence

$$b_{\lambda\rho}\beta^{\lambda\rho} = -\, b_{\lambda\rho}\varepsilon^{\lambda\gamma}\Big\{\, \varepsilon^{\alpha\rho}\Big[v|_{\alpha} + v_{\vartheta}\, b^{\vartheta}_{\alpha}\Big]\Big|_{\gamma} - \frac{\varepsilon^{\vartheta\alpha}\, v_{\vartheta}|_{\alpha}}{2}\, b^{\rho}_{\gamma}\Big\}. \tag{5_6.11}$$

Taking the second covariant differentiation of (5_3.3) and applying the Codazzi equations, we write

$$\varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}Y_{\alpha\beta}|_{\lambda\upsilon}=\varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}\left[\frac{v_\alpha|_\beta + v_\beta|_\alpha}{2}\right]\Big|_{\lambda\upsilon}-\varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}b_{\alpha\beta}v|_{\lambda\upsilon}. \quad (5\_6.12)$$

Subtracting (5_6.12) from (5_6.11), we get

$$b_{\lambda\rho}\beta^{\lambda\rho}-\varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}Y_{\alpha\beta}|_{\lambda\upsilon}= -b_{\lambda\rho}\varepsilon^{\lambda\gamma}\left\{\varepsilon^{\alpha\rho}\left[v|_\alpha+v_\vartheta b^\vartheta_\alpha\right]\Big|_\gamma - \frac{\varepsilon^{\vartheta\alpha}v_\vartheta|_\alpha}{2}b^\rho_\gamma\right\}$$

$$-\varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}\left[\frac{v_\alpha|_\beta + v_\beta|_\alpha}{2}\right]\Big|_{\lambda\upsilon}+\varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}b_{\alpha\beta}v|_{\lambda\upsilon}.$$

Simplifying

$$b_{\lambda\rho}\beta^{\lambda\rho}-\varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}Y_{\alpha\beta}|_{\lambda\upsilon}= -b^\upsilon_\rho\, a^{\gamma\eta}\varepsilon_{\upsilon\eta}\left\{\varepsilon^{\alpha\rho}\left[v_\vartheta b^\vartheta_\alpha\right]\Big|_\gamma - \frac{\varepsilon^{\vartheta\alpha}v_\vartheta|_\alpha}{2}b^\rho_\gamma\right\}$$

$$-\varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}\left[\frac{v_\alpha|_\beta + v_\beta|_\alpha}{2}\right]\Big|_{\lambda\upsilon}. \quad (5\_6.13)$$

Equation (5_6.13) contains the second covariant derivatives of covariant surface vectors and tensors, then from (3_3.61) and (3_3.62) we have

$$\varepsilon^{\lambda\beta}v_\alpha|_{\beta\lambda} = -v_\rho\, a^{\rho\lambda}\,\varepsilon_{\lambda\alpha}\,K$$
$$\varepsilon^{\alpha\upsilon}v_\beta|_{\alpha\upsilon} = v_\rho\, a^{\rho\upsilon}\,\varepsilon_{\upsilon\beta}\,K \quad (5\_6.14)$$

and

$$v_\beta|_{\alpha\lambda\upsilon}-v_\beta|_{\alpha\upsilon\lambda}=R^{\rho}_{\cdot\beta\lambda\upsilon}\,v_\rho|_\alpha + R^{\rho}_{\cdot\alpha\lambda\upsilon}\,v_\beta|_\rho \quad (5\_6.15)$$

$$= a^{\rho\xi}\,\varepsilon_{\xi\beta}\,\varepsilon_{\lambda\upsilon}K\,v_\rho|_\alpha + a^{\rho\xi}\,\varepsilon_{\xi\alpha}\,\varepsilon_{\lambda\upsilon}K\,v_\beta|_\rho.$$

Now, substituting (5_6.14) and (5_6.15) into (5_6.13), we write

$$b_{\lambda\rho}\beta^{\lambda\rho} - \varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}Y_{\alpha\beta}|_{\lambda\upsilon} = -b^{\upsilon}_{\rho}\, a^{\eta\lambda}\varepsilon_{\upsilon\eta}\left\{ \varepsilon^{\alpha\rho}\left[v_{\vartheta}b^{\vartheta}_{\alpha}\right]\Big|_{\lambda} - \frac{\varepsilon^{\vartheta\alpha}\, v_{\vartheta}|_{\alpha}}{2} b^{\rho}_{\lambda}\right\}$$

$$- \frac{1}{2}\left\{\varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}v_{\beta}|_{\alpha\upsilon} + \varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}v_{\alpha}|_{\beta\lambda} + \varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}\left[v_{\beta}|_{\alpha\lambda\upsilon} - v_{\beta}|_{\alpha\upsilon\lambda}\right]\right\}$$

$$= -b^{\upsilon}_{\rho}\, a^{\eta\lambda}\varepsilon_{\upsilon\eta}\left\{ \varepsilon^{\alpha\rho}\left[v_{\vartheta}b^{\vartheta}_{\alpha}\right]\Big|_{\lambda} - \frac{\varepsilon^{\vartheta\alpha}\, v_{\vartheta}|_{\alpha}}{2} b^{\rho}_{\lambda}\right\}$$

$$- \frac{1}{2}\left\{\varepsilon^{\lambda\beta}\left[v_{\rho}a^{\rho\upsilon}\varepsilon_{\upsilon\beta}K\right]\Big|_{\lambda} - \varepsilon^{\alpha\upsilon}\left[v_{\rho}a^{\rho\lambda}\varepsilon_{\lambda\alpha}K\right]\Big|_{\upsilon} + \varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}\left[a^{\rho\xi}\varepsilon_{\xi\beta}\varepsilon_{\lambda\upsilon}Kv_{\rho}|_{\alpha} +\right.\right.$$

$$\left.\left. + a^{\rho\xi}\varepsilon_{\xi\alpha}\varepsilon_{\lambda\upsilon}Kv_{\beta}|_{\rho}\right]\right\}$$

$$= -b^{\upsilon}_{\rho}\left\{a^{\lambda\rho}\left[v_{\vartheta}b^{\vartheta}_{\upsilon}\right]\Big|_{\lambda} - \frac{1}{2}a^{\eta\lambda}\, v_{\upsilon}|_{\eta}\, b^{\rho}_{\lambda}\right\} +$$

$$b^{\upsilon}_{\rho}\left\{a^{\chi\lambda}\,\delta^{\rho}_{\upsilon}\left[v_{\vartheta}b^{\vartheta}_{\chi}\right]\Big|_{\lambda} - \frac{1}{2}a^{\eta\lambda}\, v_{\eta}|_{\upsilon}\, b^{\rho}_{\lambda}\right\} - \left[a^{\eta\chi}\, v_{\eta}K\right]\Big|_{\chi}$$

$$= -\, b^{\lambda}_{\upsilon}\left[v^{\vartheta}\, b^{\upsilon}_{\vartheta}\right]\Big|_{\lambda} + b^{\upsilon}_{\upsilon}\left[v^{\vartheta}\, b^{\lambda}_{\vartheta}\right]\Big|_{\lambda} - \left[a^{\eta\chi}\, v_{\eta}K\right]\Big|_{\chi}$$

$$= v^{\vartheta}\left[b^{\lambda}_{\upsilon}\, b^{\upsilon}_{\vartheta}|_{\lambda} + b^{\upsilon}_{\upsilon}\, b^{\lambda}_{\vartheta}|_{\lambda} - K|_{\vartheta}\right] + v^{\vartheta}|_{\lambda}\left[b^{\lambda}_{\upsilon}\, b^{\upsilon}_{\vartheta} + b^{\upsilon}_{\upsilon}\, b^{\lambda}_{\vartheta} - K\,\delta^{\lambda}_{\vartheta}\right].$$

Using (3_3.48) and the Codazzi equations, we get finally,

$$b_{\lambda\rho}\beta^{\lambda\rho} - \varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}Y_{\alpha\beta}|_{\lambda\upsilon} = 0 \tag{5\_6.16}$$

and equation (5_6.10) becomes

$$\dot{K} = Y_{\beta\lambda}\left[b^{\lambda\rho}b^{\beta}_{\rho} - b^{\lambda\beta}b^{\rho}_{\rho}\right] + \varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}Y_{\alpha\beta}|_{\lambda\upsilon}. \tag{5\_6.17}$$

If we take the covariant differentiation of equation (5_6.2) we write

$$\text{ß}^{\alpha\beta}|_{\alpha} = \varepsilon^{\alpha\lambda}\left[\Omega^{\beta}|_{\lambda} - \Omega b^{\beta}_{\lambda}\right]\Big|_{\alpha}$$

$$= \varepsilon^{\alpha\lambda}\Omega^{\beta}|_{\lambda\alpha} - \varepsilon^{\alpha\lambda}\Omega|_{\alpha}b^{\beta}_{\lambda} - \varepsilon^{\alpha\lambda}\Omega\; b^{\beta}_{\lambda}|_{\alpha}. \qquad (5\_6.18)$$

Using the Codazzi equations and equations (3_3.61), (3_3.48) we obtain

$$\text{ß}^{\alpha\beta}|_{\alpha} = -\,\Omega_{\rho}\;\varepsilon^{\rho\beta}\;K - \varepsilon^{\alpha\lambda}\Omega|_{\alpha}b^{\beta}_{\lambda}$$

$$= -\,\Omega^{\alpha}\;a_{\rho\alpha}\;\varepsilon^{\rho\beta}\;K - \varepsilon^{\alpha\lambda}\Omega|_{\alpha}b^{\beta}_{\lambda}$$

$$= -\,\Omega^{\alpha}\;a_{\rho\alpha}\;\varepsilon^{\rho\beta}\,\frac{1}{2}\,\varepsilon_{\vartheta\upsilon}\varepsilon^{\chi\gamma}b^{\vartheta}_{\chi}\;b^{\upsilon}_{\gamma} - \varepsilon^{\alpha\lambda}\Omega|_{\alpha}b^{\beta}_{\lambda}$$

$$= -\,\Omega^{\alpha}\;a_{\rho\alpha}\;\frac{1}{2}\,\varepsilon^{\chi\gamma}\left[\,b^{\rho}_{\chi}\;b^{\beta}_{\gamma} - b^{\beta}_{\chi}\;b^{\rho}_{\gamma}\,\right] - \varepsilon^{\alpha\lambda}\Omega|_{\alpha}b^{\beta}_{\lambda}$$

$$= -\,\Omega^{\vartheta}\;a_{\rho\vartheta}\;\varepsilon^{\alpha\lambda}\;b^{\rho}_{\alpha}\;b^{\beta}_{\lambda} - \varepsilon^{\alpha\lambda}\Omega|_{\alpha}b^{\beta}_{\lambda}$$

$$= -\,\varepsilon^{\alpha\lambda}\;b^{\beta}_{\lambda}\left[\Omega^{\vartheta}\;b_{\vartheta\alpha} + \Omega|_{\alpha}\right].$$

Using (5_5.14) and (5_5.23), we get

$$\text{ß}^{\alpha\beta}|_{\alpha} = -\,\varepsilon^{\alpha\lambda}\;b^{\beta}_{\lambda}\;\overline{\Omega}_{,\alpha}\,.\mathbf{n}$$

$$= \varepsilon^{\alpha\lambda}\;b^{\beta}_{\lambda}\;\varepsilon^{\vartheta\chi}\;Y_{\vartheta\alpha}|_{\chi}$$

$$= a^{\alpha\rho}\;b^{\beta\lambda}\left[Y_{\rho\alpha}|_{\lambda} - Y_{\lambda\alpha}|_{\rho}\,\right] \qquad (5\_6.19)$$

Thus, we write finally the set of compatibility equations from (5_6.4), (5_6.16) and (5_6.19) as follows

$$\varepsilon_{\alpha\beta}\ \text{ß}^{\alpha\beta} = \varepsilon^{\alpha\lambda}\ Y_{\alpha\beta}\ b^{\beta}_{\lambda}$$

$$b_{\lambda\rho}\text{ß}^{\lambda\rho} = \varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}Y_{\alpha\beta}|_{\lambda\upsilon} \tag{5\_6.20}$$

$$\text{ß}^{\alpha\beta}|_{\alpha} = a^{\alpha\rho}\ b^{\beta\lambda}\left[Y_{\rho\alpha}|_{\lambda} - Y_{\lambda\alpha}|_{\rho}\right]$$

This set of equations comprises 4 equations with 7 unknowns, 4 components of bending strains and 3 components of membrane strains.

Equations (5_5.21) and (5_5.22) are also compatibility equations and Ω^{β} can be eliminated to form a single equation in the normal component of the angular velocity Ω. The procedure starts first by eliminating the tangential component of the angular velocity from (5_5.21). Multiplication of (5_5.21) by $\varepsilon^{\xi\eta}\varepsilon^{\rho\gamma}\ b_{\xi\rho}$, yield

$$\varepsilon^{\xi\eta}\varepsilon^{\rho\gamma}\ b_{\xi\rho}\left[\varepsilon^{\alpha\lambda}\ Y_{\alpha\gamma}|_{\lambda} + \Omega|_{\gamma}\right] = -\ \varepsilon^{\xi\eta}\varepsilon^{\rho\gamma}b_{\xi\rho}\ b_{\beta\gamma}\ \Omega^{\beta}. \tag{5\_6.21}$$

The right hand side of (5_6.21) can further be simplified in the following manner

$$\varepsilon^{\xi\eta}\ \varepsilon^{\rho\gamma}\ b_{\xi\rho}\ b_{\beta\gamma} = \varepsilon^{\xi\eta}\ \varepsilon_{\rho\gamma}b^{\rho}_{\xi}\ b^{\gamma}_{\beta}$$

$$= \varepsilon^{\rho\gamma}\ \varepsilon_{\rho\gamma}\ b^{\xi}_{\xi}\ b^{\eta}_{\beta} + \varepsilon^{\gamma\rho}\ \varepsilon_{\rho\gamma}\ b^{\eta}_{\xi}\ b^{\xi}_{\beta}$$

$$= b^{\xi}_{\xi}\ b^{\eta}_{\beta} - b^{\eta}_{\xi}\ b^{\xi}_{\beta} = b^{1}_{1}\ b^{\eta}_{\beta} + b^{2}_{2}\ b^{\eta}_{\beta} - b^{\eta}_{1}\ b^{1}_{\beta} - b^{\eta}_{2}\ b^{2}_{\beta}$$

$$\varepsilon^{\xi\eta}\ \varepsilon^{\rho\gamma}\ b_{\xi\rho}\ b_{\beta\gamma} = \delta^{\eta}_{\beta}\ K. \tag{5_6.22}$$

Hence, equation (5_6.11) becomes

$$\varepsilon^{\xi\eta}\varepsilon^{\rho\gamma}\ b_{\xi\rho}\left[\varepsilon^{\alpha\lambda}\ Y_{\alpha\gamma}|_{\lambda} + \Omega|_{\gamma}\right] = -\ \delta^{\eta}_{\beta}\ K\ \Omega^{\beta} = -\ K\ \Omega^{\eta}. \tag{5_6.23}$$

For surfaces, where K is different from zero i.e surfaces which are not developable, we write the following

$$\Omega^{\eta}|_{\eta} = -\left[\frac{\varepsilon^{\xi\eta}\varepsilon^{\rho\gamma}\ b_{\xi\rho}\left[\varepsilon^{\alpha\lambda}\ Y_{\alpha\gamma}|_{\lambda} + \Omega|_{\gamma}\right]}{K}\right]\Bigg|_{\eta}. \tag{5_6.24}$$

Using the Codazzi relations from (3_3.51), equation (5_6.24) becomes

$$\Omega^{\lambda}|_{\lambda} = -\ \varepsilon^{\xi\lambda}\varepsilon^{\rho\gamma}\ b_{\xi\rho}\left[\frac{\left[\varepsilon^{\alpha\eta}\ Y_{\alpha\gamma}|_{\eta} + \Omega|_{\gamma}\right]}{K}\right]\Bigg|_{\lambda}. \tag{5_6.25}$$

Substituting (5_6.25) into (5_5.22), we obtain a second order partial differential equation in the normal component of the angular velocity

$$\varepsilon^{\xi\lambda}\varepsilon^{\rho\gamma}\ b_{\xi\rho}\left[\frac{\left[\varepsilon^{\alpha\eta}\ Y_{\alpha\gamma}|_{\eta} + \Omega|_{\gamma}\right]}{K}\right]\Bigg|_{\lambda} + \Omega\ b^{\lambda}_{\lambda} + \varepsilon^{\alpha\lambda}\ Y_{\alpha\beta}\ b^{\beta}_{\lambda} = 0. \tag{5_6.26}$$

5_7 The static_geometric analogy

The static_geometric analogy is a theorem established between the static and geometric equations of shell theory. It relates formally the equilibrium equations comprising forces and moments , and the compatibility equations comprising the stretching and bending strains, in the absence of surface traction.

This theorem, according to Naghdi (1972), was introduced simultaneously and independently by Gol'denveizer and Lur'e in the context of the classical theory of shells under the Kirchhoff_Love hypothesis. An independent version of this theorem written in tensor notation was introduced by Sanders (1959), and reproduced in the paper by Budiansky and Sanders (1963) in their attempt to develop a first order linear shell theory. Also an analogous result was obtained by Elias (1966), using vector notation, to establish a dual formulation of thin shell theory.

Calladine (1977) extended the version of the analogy to include even normal surface traction and explained the origin and limitation of such analogy. Calladine's work was restricted to shallow shell theory, in which he splits the shell surface into two coincident surfaces called S and B carrying separately the stretching and bending stresses respectively, and introduced the change of Gaussian curvature as a prime variable.

The usual assumption in the equilibrium of shell element, according to Elias (1966), is that the moments about the normal to the surface are set equal to zero, i.e couple stress stress couples $m^{\alpha 3} = 0$.

The set of equilibrium equations in (4_4.18) when the surface is free from surface traction becomes

$$
\left.\begin{aligned}
&n^{\alpha\beta} b_{\alpha\beta} + q^{\alpha}|_{\alpha} = 0 \\
&n^{\alpha\rho}|_{\alpha} - q^{\alpha}\, b^{\rho}_{\alpha} = 0 \\
&\varepsilon_{\lambda\rho} \left[m^{\gamma\rho} b^{\lambda}_{\gamma} - n^{\lambda\rho} \right] = 0 \\
&m^{\gamma\rho}|_{\gamma} - q^{\rho} = 0
\end{aligned}\right\}. \qquad (5\_7.1)
$$

If we set

$$
\begin{aligned}
n^{\alpha\beta} &\longleftrightarrow - B^{\alpha\beta} \\
m^{\alpha\beta} &\longleftrightarrow \varepsilon^{\alpha\lambda}\, \varepsilon^{\beta\rho} Y_{\lambda\rho}
\end{aligned} \qquad (5\_7.2)
$$

and substitute the right hand side of equation (5_7.2) into the equilibrium equations (5_7.1), we obtain the compatibility equations in (5_6.20), where q^{α} is

$$
m^{\gamma\rho}|_{\gamma} = q^{\rho}. \qquad (5\_7.3)
$$

Before closing this chapter, let us write the rates of change of the normal curvature and the twist. Differentiating equations (3_3.31) with respect to time, we write

$$
\dot{k}_n = - \left[\frac{b_{\alpha\beta}\, d\vartheta^{\alpha} d\vartheta^{\beta}}{a_{\gamma\eta}\, d\vartheta^{\gamma} d\vartheta^{\eta}} \right]^{\dot{}} = - \frac{\left[\dot{b}_{\alpha\beta}\, a_{\gamma\eta} - b_{\alpha\beta}\, \dot{a}_{\gamma\eta} \right] d\vartheta^{\alpha} d\vartheta^{\beta}}{(a_{\gamma\eta})^{2}\, d\vartheta^{\gamma} d\vartheta^{\eta}} \qquad (5\_7.4)
$$

$$
\dot{\tau} = \left[\frac{b^{\lambda}_{\alpha}\, \varepsilon_{\beta\lambda} d\vartheta^{\alpha} d\vartheta^{\beta}}{a_{\gamma\eta}\, d\vartheta^{\gamma} d\vartheta^{\eta}} \right]^{\dot{}} = \frac{\left[\dot{b}^{\lambda}_{\alpha}\, \varepsilon_{\beta\lambda}\, a_{\gamma\eta} - b^{\lambda}_{\alpha}\, \varepsilon_{\beta\lambda}\, \dot{a}_{\gamma\eta} \right] d\vartheta^{\alpha} d\vartheta^{\beta}}{(a_{\gamma\eta})^{2}\, d\vartheta^{\gamma} d\vartheta^{\eta}}. \qquad (5\_7.5)
$$

CHAPTER SIX

CONSTITUTIVE EQUATIONS FOR ELASTIC SHELLS

6_1 Introduction

The constitutive equations define an additional set of shell equations and represent the third class of relations according to our previous classification. This set of equations, as defined earlier, gives the relationship between the internal forces and moments and the deformation of the reference surface. Equivalently, it gives the stress_strain relationship. Therefore the constitutive equations establish the necessary link requested for the determination of all the remaining unknowns appearing in the equilibrium and deformation of the shell surface.

Since the field equations that express the equilibrium of solids, liquids, and gases are the same in the classical continuum mechanics, it is only the constitutive equations which differ in order to describe the physical behaviour of the specific medium. Thus, the constitutive equations, even those which have the most general character, are always set to express only a particular model. It is the reason why many forms of constitutive equations can be found in the literature of shell theory depending on the problem at hand.

The basic discrepancy in all different sets of constitutive equations found in the literature of shells, is the introduction of some basic assumptions that are conform with the nature of the given model. For example, the linearity and nonlinearity of the equations could be due to, the geometric relations between strains and displacements, or the material physical property being elastic, elasto_plastic or plastic ..etc. Moreover, the mechanical property expressed by the material from which the shell is made (which can be anisotropic, orthotropic, or isotropic) offers great possibilities for further simplifications in the equations.

Natural or manufactured solids are not, in general, perfectly rigid. The action of a suitable force on these solids could bring considerable change both in size and shape. Solids made of an elastic material recover their shape and size, as soon as the force which induced the changes has ceased to act, if the changes are not too considerable.

In what follows the constitutive equations will be concerned with shells that have elastic material properties.

6_2 The rate of work of forces and stresses

In the following analysis, it is intended to keep the equations of a general character and establish a relationship between stresses and strains using a strain energy function.

If we imagine a surface having the same characteristics as the one established for the equilibrium equations, and consider a bounding curve C around the surface, the stresses are referred to the curve C and their magnitudes depend on the normal $d\,\overline{\eta}$ to the curve C, fig.(6_2.1).

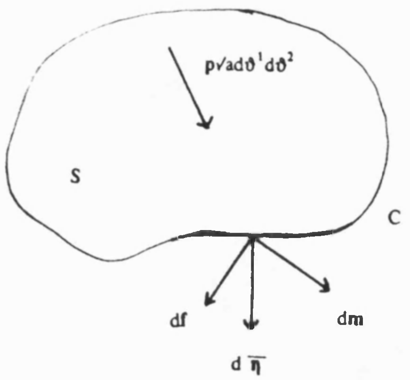


fig.(6_2.1)

The rate of work of the stress resultants, couple resultants acting on the bounding curve and the external forces applied to the surface can be derived as follows:

The rate of work of the stress resultants and the stress couples per unit length along a curve C is obtained by multiplying equations (4_3.17) and (4_3.19) by the velocity vector $\mathbf{v}$ and the angular velocity vector $\overline{\Omega}$ respectively. $\mathbf{v}$ and $\overline{\Omega}$ are defined by (5_2.2) and (5_5.11) respectively and if we set $m^{3\alpha}$ equal zero then the total rate of work being done by boundary forces and moments is equal to

$$\int_C \left[\left(n^{\alpha\lambda}\mathbf{a}_{\lambda} + q^{\alpha}\mathbf{n}\right).\mathbf{v} + \left(\varepsilon_{\lambda\gamma} m^{\alpha\gamma}\mathbf{a}^{\lambda}\right).\overline{\Omega}\right]\varepsilon_{\alpha\rho}d\vartheta^{\rho}. \qquad (6\_2.1)$$

The total rate of work being done by the applied surface loads is obtained by multiplying (4_4.1) by the velocity vector $\mathbf{v}$, i.e

$$\iint_S \mathbf{p}.\mathbf{v}\, dS \quad , \qquad \mathbf{p} = p^{\alpha}\mathbf{a}_{\alpha} + p\,\mathbf{n} \qquad (6\_2.2)$$

The sum of (6_2.1) and (6_2.2) is

$$\int_C \left[\left(n^{\alpha\lambda}\mathbf{a}_{\lambda} + q^{\alpha}\mathbf{n}\right).\mathbf{v} + \left(\varepsilon_{\lambda\gamma} m^{\alpha\gamma}\mathbf{a}^{\lambda}\right).\overline{\Omega}\right]\varepsilon_{\alpha\rho}\, d\vartheta^{\rho} + \iint_S \mathbf{p}.\mathbf{v}\, dS. \qquad (6\_2.3)$$

In order to transform the line integral into an integral over the surface, we can use Green's theorem (given in Green & Zerna (1968)), it states that;

If a vector $\mathbf{v}$ defined through a volume τ which is bounded by a surface S to which a unit normal vector $\mathbf{n}$ is erected, then the usual form of Green's theorem is

$$\iiint_\tau v^r|_r\, d\tau = \iiint_\tau \frac{1}{\sqrt{g}}\,(v^r\sqrt{g}\,)_{,r}\, d\tau = \iint_S v^r n_r\, dS$$

where

$$\mathbf{n} = n_r\, \mathbf{g}^r \quad , \qquad \mathbf{v} = v^r\, \mathbf{g}_r \qquad r = 1,2,3$$

which means, Niordson (1985), that the volume_integral of the divergence of a vector field equals the flux out of the boundaries.
The application of this integral can be extended to the case of a vector $\mathbf{v}$ applied to a surface S bounded by a curve C to which a unit normal $\mathbf{u}$ is attached, then we write

$$\iint_S v^\alpha|_\alpha \, dS = \iint_S \frac{1}{\sqrt{a}}(\sqrt{a}\ v^\alpha)_{,\alpha} \, dS = \int_c u_\alpha v^\alpha \, ds = \int_c \varepsilon_{\alpha\beta} v^\alpha d\vartheta^\beta \qquad (6\_2.4)$$

where

$$\mathbf{u} = u_\alpha \, \mathbf{a}^\alpha \quad , \qquad u_\alpha = \varepsilon_{\alpha\beta} \frac{d\vartheta^\beta}{ds}.$$

Therefore, by using (6_2.4), equation (6_2.3) becomes

$$\iint_S \left(\left[\left(n^{\alpha\lambda}\mathbf{a}_\lambda + q^\alpha \mathbf{n} \right) \sqrt{a}.\mathbf{v} + \left(\varepsilon_{\lambda\rho} m^{\alpha\rho} \mathbf{a}^\lambda \right) \sqrt{a}.\boldsymbol{\Omega} \right]_{,\alpha} + \mathbf{p}.\mathbf{v}\sqrt{a} \right) d\vartheta^1 d\vartheta^2 \qquad (6\_2.5)$$

From the equilibrium equations (4_4.6) and (4_4.15) we have

$$\left[\left(n^{\alpha\lambda}\mathbf{a}_\lambda + q^\alpha \mathbf{n} \right) \sqrt{a} \right]_{,\alpha} + \left(p^\alpha \mathbf{a}_\alpha + p\ \mathbf{n} \right) = 0$$

$$\left[\left(\varepsilon_{\lambda\rho} m^{\alpha\rho} \mathbf{a}^\lambda \right) \sqrt{a} \right]_{,\alpha} + \varepsilon_{\alpha\rho} \sqrt{a} \left(n^{\alpha\rho}\mathbf{n} - q^\alpha \mathbf{a}^\rho \right) = 0.$$

Performing the differentiation in equation (6_2.5) and adding and subtracting the quantity

$$\varepsilon_{\alpha\rho} \sqrt{a} \left(n^{\alpha\rho}\mathbf{n} - q^\alpha \mathbf{a}^\rho \right) .\boldsymbol{\Omega},$$

we obtain

$$\iint_S \left[\left(n^{\alpha\lambda}\mathbf{a}_\lambda\sqrt{a}+q^\alpha\mathbf{n}\ \sqrt{a}\right).\mathbf{v}_{,\alpha}+\left(\varepsilon_{\lambda\rho}m^{\alpha\rho}\mathbf{a}^\lambda\sqrt{a}\right).\boldsymbol{\Omega}_{,\alpha}-\varepsilon_{\alpha\rho}\sqrt{a}\left(n^{\alpha\rho}\mathbf{n}-q^\alpha\mathbf{a}^\rho\right).\boldsymbol{\Omega}\right]d\vartheta^1 d\vartheta^2.$$

From (5_5.15), we have

$$\mathbf{v}_{,\alpha} = Y_{\alpha\beta}\mathbf{a}^\beta+\boldsymbol{\Omega}\times\mathbf{a}_\alpha$$

together with some base vector relations, the rate of work becomes

$$\iint_S \left[n^{\alpha\lambda}\ Y_{\alpha\lambda}+\ \varepsilon_{\lambda\rho}m^{\alpha\rho}\mathbf{a}^\lambda.\boldsymbol{\Omega}_{,\alpha}\right]\ dS. \tag{6_2.6}$$

Manipulating equation (5_6.1), the above integral becomes

$$\iint_S \left[n^{\alpha\lambda}\ Y_{\alpha\lambda}+\ \varepsilon_{\gamma\alpha}\varepsilon_{\lambda\rho}\ m^{\alpha\rho}\beta^{\gamma\lambda}\ \right]\ dS. \tag{6_2.7}$$

Finally the rate of work in equation (6_2.7) can be written as

$$\iint_S \left[n^{\alpha\lambda}\ Y_{\alpha\lambda}+\ \ m^{\alpha\lambda}C_{\alpha\lambda}\right]\ dS \tag{6_2.8}$$

where

$$C_{\alpha\lambda} = \varepsilon_{\gamma\alpha}\varepsilon_{\rho\lambda}\ \beta^{\gamma\rho}. \tag{6_2.9}$$

The new second order covariant tensor $C_{\alpha\lambda}$ is the covariant bending strain tensor and in general

$$C_{\alpha\lambda} \neq C_{\lambda\alpha}.$$

6_3 The strain energy function

The quantity between brackets in equation (6_2.8) contains the derivatives with respect to time of the metrics and the curvature tensors. In an elastic shell the work done by couples and forces is conserved in the shell as an internal energy, usually called the elastic strain energy. Thus, if W is the strain energy per unit current area, then

$$\left(n^{\alpha\lambda}\, Y_{\alpha\lambda} + m^{\alpha\lambda} C_{\alpha\lambda} \right) = \frac{1}{\sqrt{a}} \, \frac{\partial(\sqrt{a}\; W\,)}{\partial t}. \qquad (6\_3.1)$$

Combination of (6_2.9), (5_6.6) and use of (5_3.3), we write

$$\left. \begin{aligned} C_{\alpha\lambda} &= - \, Y_{\alpha\rho} b^{\rho}_{\lambda} + \dot{b}_{\alpha\lambda} \\ Y_{\alpha\rho} &= \frac{1}{2}\, \dot{a}_{\alpha\rho} \end{aligned} \right\}. \qquad (6\_3.2)$$

If we assume that W is a unique function of $a_{\alpha\beta}$, $b_{\alpha\beta}$ and the coordinates, ϑ^{α}, then we can apply the chain rule to the right hand side of (6_3.1) and get

$$\frac{1}{\sqrt{a}} \left(\frac{\partial(\sqrt{a}\; W\,)}{\partial a_{\alpha\lambda}} \, \frac{\partial a_{\alpha\lambda}}{\partial t} + \frac{\partial(\sqrt{a}\; W\,)}{\partial b_{\alpha\lambda}} \, \frac{\partial b_{\alpha\lambda}}{\partial t} \right)$$

$$= \frac{1}{\sqrt{a}} \left(\frac{\partial(\sqrt{a}\; W\,)}{\partial a_{\alpha\lambda}} \, \dot{a}_{\alpha\lambda} + \frac{\partial(\sqrt{a}\; W\,)}{\partial b_{\alpha\lambda}} \, \dot{b}_{\alpha\lambda} \right). \qquad (6\_3.3)$$

Now, using (6_3.2) and (6_3.3), equation (6_3.1) becomes

$$\frac{1}{2} \dot{a}_{\alpha\lambda} \left(n^{\alpha\lambda} - m^{\alpha\rho} b^{\lambda}_{\rho} \right) + m^{\alpha\lambda} \dot{b}_{\alpha\lambda} =$$

$$= \frac{1}{\sqrt{a}} \left(\frac{\partial(\sqrt{a}\ W)}{\partial a_{\alpha\lambda}} \dot{a}_{\alpha\lambda} + \frac{\partial(\sqrt{a}\ W)}{\partial b_{\alpha\lambda}} \dot{b}_{\alpha\lambda} \right). \qquad (6\_3.4)$$

Differentiating (6_3.4) with respect to $\dot{b}_{\alpha\lambda}$, $\dot{a}_{\alpha\lambda}$ we write the following constitutive equations for stress couples and membrane stresses

$$m^{(\alpha\lambda)} = \frac{1}{\sqrt{a}} \frac{\partial(\sqrt{a}\ W)}{\partial b_{\alpha\lambda}}$$

$$n^{\alpha\lambda} - m^{\alpha\rho} b^{\lambda}_{\rho} = \frac{2}{\sqrt{a}} \frac{\partial(\sqrt{a}\ W)}{\partial a_{\alpha\lambda}} \qquad (6\_3.5)$$

where $m^{(\alpha\lambda)}$ stand for only the symmetric part of $m^{\alpha\lambda}$ due to the symmetry of the curvature tensor.

From (4_4.18) the quantity $(m^{\gamma\alpha} b^{\lambda}_{\gamma} - n^{\alpha\lambda})$ is a symmetric in_plane stress tensor, we can write then

$$\left(m^{\gamma\alpha} b^{\lambda}_{\gamma} - n^{\alpha\lambda} \right) = n^{*\lambda\alpha} = n^{*\alpha\lambda}. \qquad (6\_3.6)$$

Every second order tensor may be expressed as the sum of a symmetric and skew_symmetric tensor, then

$$n^{\alpha\beta} = n^{(\alpha\beta)} + n^{[\alpha\beta]} \quad , \qquad m^{\alpha\beta} = m^{(\alpha\beta)} + m^{[\alpha\beta]} \qquad (6\_3.7)$$

where

$$n^{(\alpha\beta)} = \frac{1}{2}\left(n^{\alpha\beta} + n^{\beta\alpha}\right), \quad n^{[\alpha\beta]} = \frac{1}{2}\left(n^{\alpha\beta} - n^{\beta\alpha}\right)$$

$$m^{(\alpha\beta)} = \frac{1}{2}\left(m^{\alpha\beta} + m^{\beta\alpha}\right), \quad m^{[\alpha\beta]} = \frac{1}{2}\left(m^{\alpha\beta} - m^{\beta\alpha}\right). \tag{6_3.8}$$

The constitutive equations in (6_3.5), with the aid of (6_3.6), becomes

$$m^{(\alpha\lambda)} = \frac{1}{\sqrt{a}} \frac{\partial(\sqrt{a}\ W)}{\partial b_{\alpha\lambda}}$$

$$n^{*\alpha\lambda} = -\frac{2}{\sqrt{a}} \frac{\partial(\sqrt{a}\ W)}{\partial a_{\alpha\lambda}}. \tag{6_3.9}$$

Up to this stage, the shell equations are completely general and no assumptions have been made whatsoever. However, the ten unknown stresses and the three unknown displacements make a set of thirteen unknowns, while we have only six equilibrium equations and six constitutive equations. For the determination of the remaining unknown, we follow the work of Naghdi (1972) for the treatment of the special case of the Cosserat surface known as the restricted theory that bears on the classical theory of shells.

In order to provide a determinate theory, Naghdi (1972) assumed that the indeterminate constitutive equation for the antisymmetric part of the moment stress can be set as

$$m^{[\alpha\beta]} = 0. \tag{6_3.10}$$

Using (6_3.6), (6_3.7), (6_3.10) and (6_3.9) , $n^{\alpha\lambda}$ is

$$n^{\alpha\lambda} = \frac{1}{\sqrt{a}}\left(2\,\frac{\partial(\sqrt{a}\ W\ )}{\partial a_{\alpha\lambda}} + \frac{\partial(\sqrt{a}\ W\ )}{\partial b_{\alpha\gamma}}\, b^{\lambda}_{\gamma}\right). \qquad (6\_3.11)$$

From (6_3.8) and (6_3.11) the anti_symmetric part $n^{[\alpha\lambda]}$is

$$n^{[\alpha\lambda]} = \frac{1}{2}\left(m^{(\alpha\gamma)}b^{\lambda}_{\gamma} - m^{(\lambda\gamma)}b^{\alpha}_{\gamma} \right). \qquad (6\_3.12)$$

Lastly, equations (6_3.10) and (6_3.12) render the theory determinate, i.e. the number of unknowns is equal to the number of equations.

Equations (4_4.18) with the effect of moments about the normal discarded, becomes

$$\left.\begin{aligned} &n^{\alpha\beta}b_{\alpha\beta} + q^{\alpha}|_{\alpha} + p = 0 \\ &n^{\alpha\rho}|_{\alpha} - q^{\alpha}\, b^{\rho}_{\alpha} + p^{\rho} = 0 \\ &\varepsilon_{\lambda\rho}\left[m^{\gamma\lambda}b^{\rho}_{\gamma} - n^{\lambda\rho}\right] = 0 \\ &m^{\gamma\rho}|_{\gamma} = q^{\rho} \end{aligned}\right\}. \qquad (6\_3.13)$$

Substituting the value of the shearing forces from the last equation into the two first equations, we write

$$\left.\begin{aligned}
&n^{\alpha\beta}b_{\alpha\beta} + m^{\gamma\alpha}|_{\gamma\alpha} + p = 0\\
&n^{\alpha\rho}|_{\alpha} - m^{\gamma\alpha}|_{\gamma}\, b^{\rho}_{\alpha} + p^{\rho} = 0\\
&\varepsilon_{\lambda\rho}\left[m^{\gamma\lambda}b^{\rho}_{\gamma} - n^{\lambda\rho}\right] = 0\\
&m^{\gamma\alpha}|_{\gamma} = q^{\alpha}
\end{aligned}\right\}. \tag{6_3.14}$$

If we substitute (6_3.8), (6_3.12) into the first two equations of (6_3.14) and make use of (6_3.10), we have

$$n^{(\alpha\rho)}|_{\alpha} + \frac{1}{2}\left(m^{(\alpha\gamma)}b^{\rho}_{\gamma}\right)|_{\alpha} - \frac{1}{2}\left(m^{(\rho\gamma)}b^{\alpha}_{\gamma}\right)|_{\alpha} - m^{(\gamma\alpha)}|_{\gamma}b^{\rho}_{\alpha} + p^{\rho} = 0$$

$$n^{(\alpha\lambda)}b_{\alpha\lambda} + m^{(\gamma\alpha)}|_{\gamma\alpha} + p = 0\ . \tag{6_3.15}$$

These differential equations involves only the symmetric part of the stress resultants and stress couples. The system of equations (5_3.3), (6_3.9), (6_3.10), (6_3.12), $(6\_3.14)_4$ and (6_3.15) form a determinate theory for the unknowns $n^{\alpha\beta}$, $m^{\alpha\beta}$, q^{ρ}, v^{α}, v. A brief discussion of the nature of boundary conditions and the solutions of this system of equations will make the subject of the next chapter.

CHAPTER SEVEN

BOUNDARY CONDITIONS AND SOLUTION OF EQUATIONS

7_1 Introduction

In mathematics, it is usual to discuss the solutions of any set of partial differential equations that describe a particular problem with the use of certain boundary or initial conditions depending on the physical problem. The resulting mathematical problem constitute the well known boundary value problem.

As a consequence of certain relations between forces, moments and displacements at the supporting edges, the full analysis of shell theory is completed only when the appropriate boundary conditions are attributed to the structure.

It seems that it is more convenient, while dealing with the boundary conditions, to distinguish between two types of structure. The first type concerns the closed shells, i.e. complete shells, in which clearly the concept of boundary conditions loses its meaning. However, certain conditions concerning the periodicity of the solutions of stresses and displacement must be fulfilled.

The second type of structures concerns open shells, i.e. shells with boundaries. In this category of structure, knowing the number and nature of boundary conditions is necessary. Thus, for our two dimensional set of equations derived previously special boundary conditions need to be determined, and this will constitute the task of the following section.

7_2 Boundary conditions

The nature of boundary conditions in the present work can be defined using the previous energy equation used for the determination of the constitutive equations. The rate of work of forces and moments per unit length along the boundary curve C to which the normal $d\overline{\eta}$ is attached is given by summing (4_3.14), (4_3.15) multiplied by the velocity vector **v** and (4_3.18) multiplied by the angular velocity vector $\overline{\Omega}$, i.e.

$$F = \int_c (n^{\alpha\lambda}\mathbf{a}_\lambda + q^\alpha \mathbf{n}).\mathbf{v}\, d\eta_\alpha + \int_c (\varepsilon_{\lambda\rho} m^{\alpha\rho}\mathbf{a}^\lambda).\,\overline{\Omega}\, d\eta_\alpha. \qquad (7\_2.1)$$

From equation (4_3.13) we have

$$d\eta_\alpha = \varepsilon_{\alpha\gamma} d\vartheta^\gamma$$

and the velocity and angular velocity vectors are given by

$$\mathbf{v} = v_\rho\, \mathbf{a}^\rho + v\, \mathbf{n} \quad , \quad \overline{\Omega} = \Omega^\rho \mathbf{a}_\rho + \Omega\, \mathbf{n}.$$

Substituting the above expressions into (7_2.1), we get

$$F = \int_c \left[n^{\alpha\rho} v_\rho + q^\alpha v + m^{\alpha\gamma}\left(v|_\gamma + v_\rho b^\rho_\gamma \right)\right] \varepsilon_{\alpha\gamma} d\vartheta^\gamma \qquad (7\_2.2)$$

where use of (5_5.6) has been made. It also can be further reduced using $(6\_3.14)_4$ to

$$F = \int_C \left(\left(n^{\alpha\rho} + m^{\alpha\gamma} b^{\rho}_{\gamma} \right) v_{\rho} + m^{\gamma\alpha}|_{\gamma} v + m^{\alpha\gamma} v|_{\gamma} \right) \varepsilon_{\alpha\beta} d\vartheta^{\beta} \qquad (7\_2.3)$$

Using equations $(6\_2.4)_3$ and (6_3.10), equation (7_2.3) becomes

$$F = \int_C \eta_{\alpha} \left[\left(n^{\alpha\rho} + m^{(\alpha\gamma)} b^{\rho}_{\gamma} \right) v_{\rho} + m^{(\gamma\alpha)}|_{\gamma} v + m^{(\alpha\gamma)} v|_{\gamma} \right] ds. \qquad (7\_2.4)$$

The quantity $v|_{\gamma}$ in equation (7_2.4) can be resolved into two components normal and tangential to the boundary curve C as follows, fig. (7_2.1)

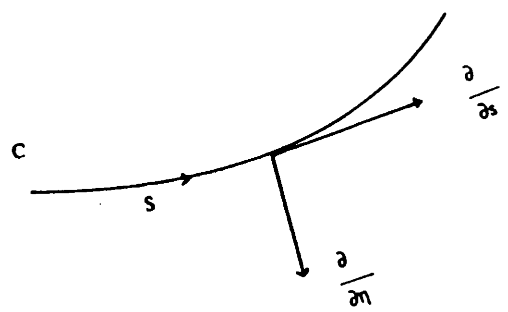


fig.(7_2.1) The boundary curve C

If $\frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial s}$ are the directional derivatives along the normal and the tangent to the curve C respectively, then

$$v|_{\gamma} = v_{,\gamma} = \eta_{\gamma} \frac{\partial v}{\partial \eta} + \tau_{\gamma} \frac{\partial v}{\partial s} \qquad (7\_2.5)$$

where τ_{γ} is given by

$$\tau_\gamma = \varepsilon_{\rho\gamma}\eta^\rho .$$

Substituting the contents of equation (7_2.5) into equation (7_2.4), we write

$$F=\int_c \left[\left(n^{\alpha\rho}+ m^{(\alpha\gamma)}b^\rho_\gamma\right)\eta_\alpha v_\rho + m^{(\gamma\alpha)}\,|_\gamma \eta_\alpha v + m^{(\alpha\gamma)}\left(\eta_\gamma\eta_\alpha \frac{\partial v}{\partial\eta} + \varepsilon_{\rho\gamma}\eta_\alpha\eta^\rho \frac{\partial v}{\partial s}\right)\right]ds. \tag{7\_2.6}$$

Performing integration by parts on the third term of the integral, then

$$\int_c \left[m^{(\alpha\gamma)}\left(\eta_\gamma\eta_\alpha \frac{\partial v}{\partial\eta} + \varepsilon_{\rho\gamma}\eta_\alpha\eta^\rho \frac{\partial v}{\partial s}\right)\right]ds =$$

$$\int_c m^{(\alpha\gamma)}\eta_\gamma\eta_\alpha \frac{\partial v}{\partial\eta}\,ds + \int_c m^{(\alpha\gamma)}\,\varepsilon_{\rho\gamma}\eta_\alpha\eta^\rho dv$$

$$=\int_c m^{(\alpha\gamma)}\eta_\gamma\eta_\alpha \frac{\partial v}{\partial\eta}\,ds + [\, m^{(\alpha\gamma)}\varepsilon_{\rho\gamma}\eta_\alpha\eta^\rho v\,] - \int_c v\,\frac{\partial}{\partial s}(m^{(\alpha\gamma)}\varepsilon_{\rho\gamma}\eta_\alpha\eta^\rho)ds$$

$$=\int_c m^{(\alpha\gamma)}\eta_\gamma\eta_\alpha \frac{\partial v}{\partial\eta}\,ds - \int_c v\,\frac{\partial}{\partial s}(m^{(\alpha\gamma)}\varepsilon_{\rho\gamma}\eta_\alpha\eta^\rho)ds. \tag{7\_2.7}$$

where we have put

$$[\, m^{(\alpha\gamma)}\varepsilon_{\rho\gamma}\eta_\alpha\eta^\rho v\,] = 0$$

since v is single_valued and the integral is taken all around C.

Now, substituting the value of the third term from (7_2.7), equation (7_2.6) becomes

$$F=\int_{c}\left[T^{\rho}v_{\rho} + T\ v + H\,\frac{\partial v}{\partial \eta}\right]ds \qquad (7\_2.8)$$

where

$$T^{\rho}=\left(n^{\alpha\rho}+ m^{(\alpha\gamma)}b^{\rho}_{\gamma}\right)\eta_{\alpha}\ , \quad T= m^{(\gamma\alpha)}|_{\gamma}\eta_{\alpha} - \frac{\partial(\ m^{(\alpha\gamma)}\varepsilon_{\rho\gamma}\eta_{\alpha}\eta^{\rho})}{\partial s}$$

$$H = m^{(\alpha\gamma)}\eta_{\gamma}\eta_{\alpha}. \qquad (7\_2.9)$$

The expressions in equation (7_2.8) show the nature of boundary conditions that are necessary for our previous set of equations. They consist of an edge stress resultant along the base vectors of the surface, a shear force resultant along the normal to the surface and finally a tangential couple to the boundary. Equation (7_2.9) shows the modified force and couple boundary conditions.

When, the investigation is carried out from the point of view of the stress solutions, four boundary conditions seem to determine completely the state of stress. These are two in_plane stresses, one shearing normal force and one couple stress. The twisting moments on the boundary which normally constitute the fifth condition, is replaced by a distributed tangential and transverse forces, combined in the expressions of the shearing normal and in_plane stresses.

The boundary conditions of shells are usually formulated from a combination of forces, couples, displacements and rotations, and not only from stresses.

7_3 Solutions of equations

If the strain energy per unit current area in (6_3.3) is a given function of $a_{\alpha\beta}$, $b_{\alpha\beta}$ and ϑ^{α} , in (6_3.5) we would expect well defined relations between stress resultants and stress couples on one hand and the membrane and bending strains in the other hand. The definition of the strain energy then, would be used in writing all the components of membrane and bending stresses in terms of three new stress functions say, the vector stress function $\Psi_{\alpha}(\vartheta^1,\vartheta^2)$ and the scalar function Φ $(\vartheta^1,\vartheta^2)$. This is consequence of the previously defined static_geometric theorem established between the homogeneous statical problem and the geometric problem of shell theory. The procedure to find these stress functions is as follows:

a)_ First, we start by writing the membrane stress tensor in a structure similar to that established for the rate of membrane strain tensor in (5_3.3). However, the tangential and normal components of the displacement vector in (5_3.3) are replaced respectively by the new vector stress function Ψ_{α} and the scalar function Φ.

b)_Secondly, if we substitute the expressions of the components of the angular velocity from (5_5.4) and (5_5.6) into the expression of the bending tensor in (5_6.2), we get an expression in the components of displacement for the bending tensor. Then, we can write a similar expression for the bending stresses, where the displacement components are replaced by the new stress functions.

As it is proved before, the equations (5_3.3) and (5_6.2) satisfy the compatibility set of equations in (5_6.20) and the equilibrium equations have the same structure as the compatibility equations, then the two static tensor equations derived above would satisfy the set of equilibrium equations.

Now, if these two static tensor equations are substituted in the constitutive equations, the strains components will be functions of the new stress functions. If the latter equations are introduced into the compatibility equations, then a system of partial differential equations in the new stress function is obtained.

However, the partial differential equations obtained by the above procedure satisfy only a structure which is free from surface traction, i.e the satisfaction of the statical homogeneous problem of shell theory.

In texts of shell theory (Novozhilov (1959), Gol'denveizer (1961), Kraus (1967) and Niordson (1985)) the mathematical problem of solving the general equations of shell theory, previously determined, can be achieved by means of two procedures. The first procedure is called the displacement method, where the fundamental unknowns are the components of displacement. the second procedure is known as the force method, where the fundamental unknowns are the stresses. In both methods the resulting system of equations is of order eight which is too formidable for an analytical solution to be possible. Consequently the analysis of a practical problems is always sought in a simpler way according to the possible simplifications offered by the structure. However, for complicated geometric structures or for cases where the surface loads vary abruptly, numerical methods such as finite elements are usually used for approximate solutions, Davies (1980).

Among the simplified shell theories that are applied to a specific cases, we mention the introduction of the principle of complex variables, Novozhilov (1964), Sanders (1969). Also the eighth order character of the equations is simplified to two simultaneous fourth order equations in terms of stress and displacement functions in the case of shallow shell theory, Vlasov (1951). Moreover the special case of shells of revolution including the cylindrical shell have now a special simplified equations for their analysis. Lastly the dominance of either stretching or bending stresses in the structure leads to the well known simplified theories of membrane and inextensional deformations respectively.

CHAPTER EIGHT

MEMBRANE THEORY AND THE INEXTENSIONAL DEFORMATION

8_1 Introduction

In the present chapter we will introduce the simplified equilibrium equations produced when bending and twisting moments are zero. This is known as the membrane theory of shells. We will also discuss the kinematic relationships produced when a shell deforms by bending with no stretching. This is known as inextensional deformation.

The two topics are intimately related since if it is possible for a shell to undergo inextensional deformation, then the shell cannot carry all load distributions by membrane action alone.

8_2 Membrane theory

Membrane theory is a spectacular simplification in shell theory. It has preceded in practice all other simplified approaches, because of its simple principle and in some cases accuracy of results.

In view of their shapes, shells have permitted engineers and designers to take a first step in the analysis of such structures, by considering the totality of loads being carried by membrane forces only. As a result of this simplification, a lowering in the order of the equations is obtained, and consequently only two boundary conditions have to be specified at every edge.

The reason that designers related the idea of membrane theory (or the so_called momentless theory) to the shape of the structures, is apparent when comparing the properties of shells to those of curved beams. Curved beams are usually designed not only to work as a compression members, but also as a flexural members. But if it is desired, one can, by specifying their shapes, the way that are supported, and the manner of its loading, eliminate the bending effect.

8_2.1 The mechanism of membrane theory

In their action and analysis, membrane shell structures are said to be analogous to those of triangulated structural trusses. In the latter, the joints are made frictionless and the external load is applied to them. As a result of this arrangement the structure will not be subject to any form of bending. Thus, all loads are carried by in_plane compressive, and tensile stresses.

The membrane hypothesis neglects the effect of shearing forces and bending moments. It only considers the applied loading totally carried by in_plane stress resultants.

It should be emphasized that in structural trusses there are three types of structures. This corresponds to whether the structure is statically determinate, statically indeterminate or a mechanism, Asplund (1966).

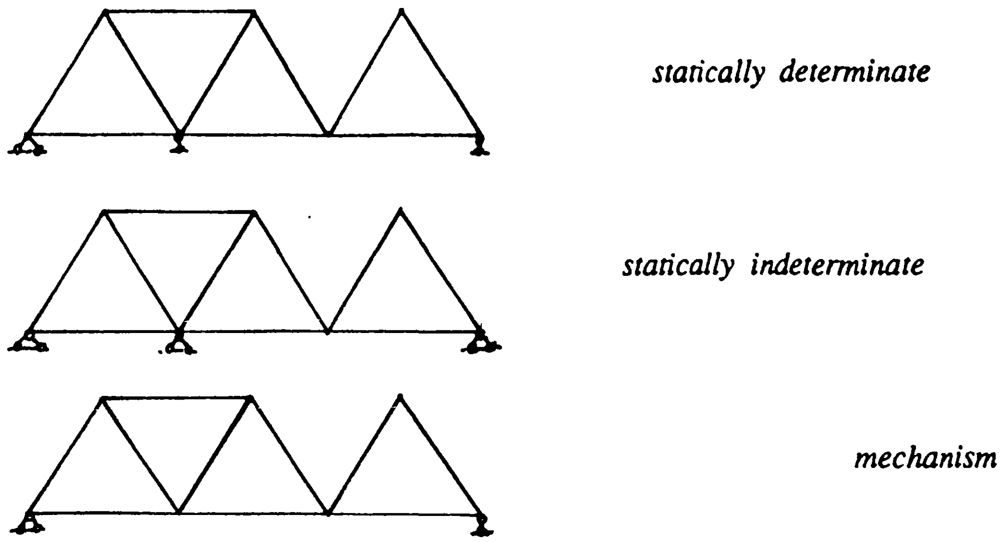


fig.(8_2.1) Types of structural trusses

Calladine (1983) has stated "..This method is known as the membrane *hypothesis*. It is a hypothesis in the normally accepted sense of the word:if it should transpire in a given case that the deflected form of the shell calculated on this basis involves changes of curvature corresponding to substantial bending_stress resultants, then the hypothesis must be abandoned, and a new, more complete analysis must be attempted. In this respect it closely resembles the *pin_joint hypothesis* for the analysis of trusses.".

The hypothesis of membrane theory is usually adopted according to one of the following reasons : either the shell has very small bending stiffness or when the changes of curvature and twist are negligible.

The first case is concerned with thin pliable skin, as it is described by Novozhilov (1959). He stated "..An absolutely flexible shell (for example, made from cloth) is not able to sustain compression forces, since any compression, however small, will cause loss of stability of its shape, i.e, the formation of wrinkles.". This kind of structure is the origin of the name "membrane" which implies that only a momentless state of stress is possible. In practice such structures are called tensile structures, where only tensile stresses can be found at any section along it. They are used for instance to construct hot_air balloons, parachutes, and airship, etc..

Structures with finite bending stiffness, on the other hand, which represent the second category, are structures where the membrane state of stress represents only one of the possible stress conditions. This category of structure is used worldwide in the construction field and industry and has shown a greater usefulness. For such structures the applicability of membrane hypothesis usually is subject to a number of conditions which must be fulfilled, concerning the shape of the structure, the type of loading, and the nature of boundary conditions.

8_2.2 The validity of the membrane theory

The applicability of the membrane theory has always been subject to some restrictions concerning, the geometry of the surface of the shell, the nature of the applied external loads,

and the attachment of the boundary supports. A number of authors such as Gol'denveizer (1961) and Novozhilov (1964) have already mentioned these restrictions. In his text book Gol'denveizer (1961) assumes that, the membrane theory is best considered only at regions away from lines of distortions of the state of stress.
To the lines of distortion belong," a) the edges of shell; b) lines along which occur discontinuities of the components of the external surface loads or of certain of their derivatives; c) lines along which the middle surface of a shell has a break or the curvature of the middle surface changes abruptly; d) lines along which the rigidity of a shell or its thickness undergoes sudden changes.". In addition to these, the membrane structure may neither be loaded along the boundaries by transverse forces nor by bending moments. Also the angle of twists and the normal displacement can not be constrained.

While we are investigating the validity of membrane theory, it appears that there is a great need to the possibility of knowing a priori whether a shell structure, can carry forces only by membrane stresses. It is very difficult to tell a priori what sort of state of stress the structure is subject to. In this context, Pavlovic (1978) has constructed a flow_chart through which a verification of the validity may be carried out, and from which the following is extracted.

If the problem is statically determinate, the stresses are determined from the static equilibrium equations of membrane theory. The corresponding strains are obtained through the constitutive relations eg. Hook's law. On the other hand if the problem of membrane theory is statically indeterminate, both the stresses and strains have to be combined together. The third step consists of, obtaining the changes of curvature and twist through the in_plane distribution of strains, and hence the bending and twisting moments are evaluated to check the flexural action. If the flexural action is found to be negligible, then the membrane hypothesis is justified.

If it is possible to carry certain loads by membrane action only, then this state of stress can be used in the lower_bound or "safe" theorem of plasticity, (provided that buckling does not occur). Heyman (1977) has used the lower_bound theorem in studying masonry arches and vaults.

8_2.3 Equilibrium of the membrane shell

The equilibrium of the membrane shell can be obtained either, directly from the principles of the hypothesis, as has been given in Timoshenko and Woinowsky_Kreiger (1959), or by simplifying the equations of the general thin shell theory.

In view of the smallness of the changes of curvature and twist, the membrane theory is obtained by neglecting the terms containing moments $m^{\alpha\beta}$ and $m^{3\alpha}$ from the equations of the general bending theory. Thus, from equations (4_4.16) and (4_4.17), we get respectively:

$$\left.\begin{aligned} n^{12} &= n^{21} \\ q^{1} = q^{2} &= 0 \end{aligned}\right\}. \tag{8_2.1}$$

Using the second equality of (8_2.1) in equation (4_4.7) we end up with the normal equation written in the following manner

$$n^{\alpha\lambda}\, b_{\lambda\alpha} + p = 0. \tag{8_2.2}$$

The equilibrium equations in the tangential directions are obtained by substituting also the second equation of (8_2.1) in equation (4_4.8), hence we write

$$n^{\alpha\beta}|_{\alpha} + p^{\beta} = 0. \tag{8_2.3}$$

We write finally the set of equilibrium equations of a membrane shell as follows

$$\left.\begin{aligned} n^{\alpha\beta} &= n^{\beta\alpha} \\ n^{\alpha\beta}|_{\alpha} + p^{\beta} &= 0 \\ n^{\alpha\lambda}\, b_{\lambda\alpha} + p &= 0 \end{aligned}\right\}. \tag{8_2.4}$$

These relations can also be obtained directly from the first principles of the membrane theory, by omitting in the equation (4_3.17) the term containing the shearing forces. In this situation the element ABCD, fig. (8_2.2), will only be subject to the simplified stress vector **df**, and **dm** = 0, i.e

$$df = n^{\alpha\lambda}\ \varepsilon_{\alpha\rho}\ d\vartheta^{\rho}\ a_{\lambda}. \tag{8_2.5}$$

The two equations (8_2.2) and (8_2.3) with the help of the symmetry condition of the shearing in_plane stresses $n^{12} = n^{21}$, are sufficient to determine the unknown stresses n^{12}, n^{11} and n^{22} provided that the boundary conditions are suitable. It is worth mentioning that the normal stress equation of the equilibrium is not differential equation. However, it is used to eliminate one unknown from the set of equilibrium equations. On the other hand, the tangential equations are differential equations.

Therefore the state of stresses in the membrane theory of shells is completely determined by the equations of equilibrium, assuming that the geometry and the external loads are known, and then the structure is statically determinate. Indeterminacies arise only from support conditions.

The solution of the problem of equilibrium of membrane shells can be split into two parts. The first part concerns a solution to satisfy the external loads i.e. the particular integral. Whereas the second parts concerns a solution which satisfy the homogeneous equation. The combined solution is the final solution which satisfies the boundary conditions.

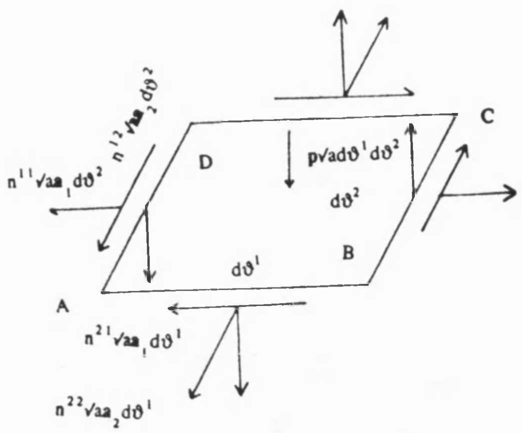


fig (8_2.2) Equilibrium of a membrane surface element

The equilibrium equation (4_4.6) without the application of the external load and shearing forces represents, the surface element ABCD subject to the vector force in (8_2.5). It is the equilibrium of a membrane shell element, loaded only at its boundaries, then (4_4.6) becomes

$$\frac{\partial}{\partial\, \vartheta^{\alpha}} \left(n^{\alpha\lambda}\sqrt{a}\; a_{\lambda} \right) = 0. \tag{8\_2.6}$$

Equation (8_2.6) represents the complementary function of the membrane theory, the solution of which satisfies the homogeneous problem of membrane theory.

8_2.4 The reciprocal surface

An interesting result based on the equilibrium of prestressed surfaces has been obtained by Williams (1986). This result consists of expressing the state of stress in homogeneous problem in terms of the geometry of a second surface called the reciprocal surface.

In order to introduce this new notion, let us try to analyse equation (8_2.6), replacing the term in bracket by an equivalent vectors $\mathbf{H}^{\alpha}$, then we write

$$\frac{\partial \left(\mathbf{H}^{\alpha} \right)}{\partial \vartheta^{\alpha}} = 0. \tag{8\_2.7}$$

Equation (8_2.7) can be written in the following manner

$$\frac{\partial}{\partial \vartheta^{1}} \left(\mathbf{H}^{1} \right) + \frac{\partial}{\partial \vartheta^{2}} \left(\mathbf{H}^{2} \right) = 0 \tag{8\_2.8}$$

which has the following solutions

$$\mathbf{H}^{1} = \frac{\partial \mathbf{Q}}{\partial \vartheta^{2}} \qquad , \qquad \mathbf{H}^{2} = - \frac{\partial \mathbf{Q}}{\partial \vartheta^{1}} \tag{8\_2.9}$$

where $\mathbf{Q}$ is the new surface which relates the functions, $\mathbf{H}^{1}$and $\mathbf{H}^{2}$.

Now we can write

$$\mathbf{H}^{\alpha} = e^{\alpha\rho} \frac{\partial \mathbf{Q}}{\partial \vartheta^{\rho}} \tag{8\_2.10}$$

where $\qquad e^{\alpha\rho} = \pm 1.$

Then, from equations (8_2.7) and (8_2.10), we write

$$n^{\alpha\beta} \sqrt{a}\, \mathbf{a}_{\beta} = e^{\alpha\rho} \mathbf{Q}_{,\rho} \tag{8\_2.11}$$

which is two equations in α

$$n^{1\beta}\sqrt{a}\,\mathbf{a}_\beta = \mathbf{Q}_{,2} \quad , \quad n^{2\beta}\sqrt{a}\,\mathbf{a}_\beta = -\,\mathbf{Q}_{,1}$$

or
$$\mathbf{Q}_{,\rho} = \varepsilon_{\gamma\rho}\, n^{\gamma\beta}\, \mathbf{a}_\beta. \tag{8_2.12}$$

That is to say

$$n^{\gamma\beta} = \varepsilon^{\gamma\rho}\, \mathbf{Q}_{,\rho}.\mathbf{a}^\beta. \tag{8_2.13}$$

The symmetry condition in the first equation of (8_2.4) implies

$$\varepsilon_{\gamma\rho}\, n^{\gamma\rho} = 0. \tag{8_2.14}$$

Therefore, scalar multiplying (8_2.12) by $\mathbf{a}^\rho$, yields

$$\mathbf{Q}_{,\rho}.\mathbf{a}^\rho = 0. \tag{8_2.15}$$

Also scalar multiplying (8_2.12) by $\mathbf{n}$ yields

$$\mathbf{Q}_{,\rho}.\mathbf{n} = 0. \tag{8_2.16}$$

Both equations (8_2.15) and (8_2.16) express the conditions of equilibrium of the unloaded new surface, they are

$$\left.\begin{aligned} \mathbf{Q}_{,\rho}\, \mathbf{n} &= 0 \\ \mathbf{Q}_{,\rho}\, \mathbf{a}^\rho &= 0 \end{aligned}\right\}. \tag{8_2.17}$$

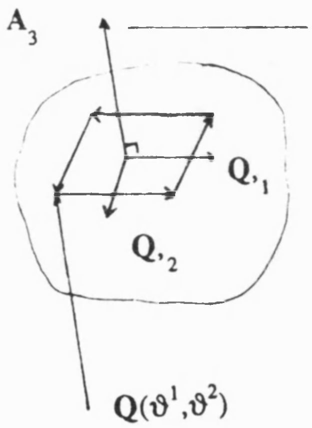
Substituting (8_2.10) into (8_2.7) and comparing to (8_2.6), we write

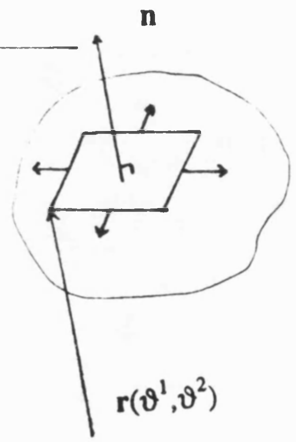
$$\frac{\partial}{\partial \vartheta^{\alpha}} \left(n^{\alpha\beta}\sqrt{a}\ \mathbf{a}_{\beta} \right) = \frac{\partial}{\partial \vartheta^{\alpha}} \left(e^{\alpha\rho}\ \mathbf{Q},_{\rho} \right) = 0 \qquad (8\_2.18)$$

and therefore,

$$e^{\alpha\rho}\ \mathbf{Q},_{\rho\alpha} = 0 \ , \qquad \mathbf{Q},_{12} - \mathbf{Q},_{21} = 0. \qquad (8\_2.19)$$

Equation (8_2.19) represents, the equilibrium of the new $\mathbf{Q}(\vartheta^1,\vartheta^2)$ surface, which is equivalent to the unloaded $\mathbf{r}(\vartheta^1,\vartheta^2)$ surface. The two surfaces are shown in fig. (8_2.3) below





a) The new **Q** *reciprocal surface* *b)* **r** *surface*

fig (8_2.3)

Substituting (8_2.12) into (8_2.5), we obtain

$$d\mathbf{Q} = \mathbf{Q},_{\rho}\ d\vartheta^{\rho} = d\mathbf{f} \qquad (8\_2.20)$$

In his work, Williams (1986), (**R** stands for **Q** in the present work), has stated that " The new surface **R** $(\vartheta^1,\vartheta^2)$ represents the state of stress in the surface $\mathbf{r}(\vartheta^1,\vartheta^2)$ in that the force across the cut along $\mathbf{a}_\alpha \; d\vartheta^\alpha$ on $\mathbf{r}$ is represented in both magnitude and direction by the vector $\mathbf{A}_\alpha d\vartheta^\alpha$ $(=\mathbf{R},_\alpha d\vartheta^\alpha)$ on $\mathbf{R}$. ". The corresponding quantities on the surface $\mathbf{Q}(\vartheta^1,\vartheta^2)$ to the surface $\mathbf{r}(\vartheta^1,\vartheta^2)$ will be represented by capital letters. Also it should be noted that the normals to both surfaces are parallel, while the directions of the base vectors are not necessarily the same, then

$$\mathbf{dn} = \mathrm{d}\mathbf{N}. \qquad (8\_2.21)$$

Therefore, from (3_3.29) and (3_3.27), we write

$$b_{\alpha\beta}\; \mathbf{a}^\alpha = B_{\alpha\beta}\; A^\alpha \qquad (8\_2.22)$$

with $\mathbf{Q},_\rho = A_\rho$. Then, scalar multiplying (8_2.22) and (8_2.12) yield

$$B_{\alpha\beta} = \varepsilon_{\gamma\alpha}\; n^{\gamma\rho}\; b_{\rho\beta} \qquad (8\_2.23)$$

$B_{\alpha\beta}$ in the above equation is not symmetric and (8_2.23) is four equations, they can be written as follows

$$B_{11} = -\sqrt{a}\left(n^{22}b_{21} + n^{21}b_{11} \right)$$

$$B_{22} = \sqrt{a}\left(n^{11}b_{12} + n^{12}b_{22} \right)$$

$$B_{12} = -\sqrt{a}\left(n^{21}b_{12} + n^{22}b_{22} \right)$$

$$B_{21} = \sqrt{a}\left(n^{11}b_{11} + n^{12}b_{21} \right).$$

Now eliminating n^{11}; n^{22} from the above set of equations and taking

$n^{12} = n^{21}$, we end up with the following

$$B_{11}b_{22} + B_{22}b_{11} = 2\, b_{12}B_{12} \tag{8_2.24}$$

Equation (8_2.24) displays the symmetry conditions between the two surfaces.

Williams (1986) concluded that, (The **Q** surface is equivalent to **R**), : "The symmetry of (8_2.24), shows that if **Q** represents a state of stress in the unloaded prestressed surface **r**, then **r** represents also a possible state of stress in the unloaded prestressed surface **Q**. There is therefore a reciprocal relationship. It should be noted that an unloaded surface can have an infinite number of different states of stress so that for any **r** there is an infinite choice of **Q** and vice versa. The two reciprocal surfaces are not, however, arbitrary since (8_2.24) must be satisfied.".

8_2.5 Conclusion

The state of membrane stress in equilibrium with certain applied loads can be considered to consist of two parts

1)_ A state of membrane stress in equilibrium with the loads, but which does not satisfy the boundary conditions.

2)_ A state of stress in equilibrium with no applied loads (except boundary forces) such that when the two states of stress are added, the boundary conditions are satisfied.

The state of stress in equilibrium with no external loads can be expressed in terms of a second surface, known as the reciprocal surface.

8_3 Inextensional deformation

The inextensional deformation of surfaces fits neatly into the kinematic aspects of deformation of curved surfaces, with the assumption that in_plane strains are equal to zero. Green & Zerna (1968) have stated " ..Shells for which no suitable membrane solution exists, their loads on the surface are not carried mainly by stress resultants, in which deformation occurs with little bending of the shell. Instead, there is considerable bending of the middle surface of the shell with little extension.."

The inextensional deformation means, deformation of the middle surface without stretching, i.e $Y_{\alpha\beta}= 0$. According to Love (1944), if the displacements of the middle surface are going to be inextensional then:

a)_ The lengths of the two line elements in the two orthogonal directions must not change when the surface is strained.

b)_ The angle between the two line elements before deformation remains the same after deformation.

As has been mentioned earlier, the way in which membrane shells carry loads is similar to that of structural trusses and the resulting structure can be statically determinate, indeterminate, or a mechanism. The structural truss is a mechanism, if it can deform without altering the length of the members. Inextensional deformation of shell structures preserves lengths on the surface, and if allowed, the structure behaves just like a mechanism. Therefore if it is desired to design a shell structure which is going to work primarily by membrane action, then one should prevent the structure from behaving as a mechanism. This will be equivalent to preventing the inextensional modes of deformation.

The aim of this theory in the present context, is to see why and when this kind of deformation may occurs and in which way the shell should be supported to avoid it.

If we put $Y_{\alpha\beta}= 0$, then from (5_3.3) we have

$$\dot{a}_{\alpha\beta}= 0. \tag{8\_3.1}$$

This is equivalent to saying that in the inextensional mode of deformation, the deformation consists only of small flexures of the surface and does not contain extensions or contraction and in_plane shear deformation.

From (5_2.8) we express the rate of change of the membrane strains as functions of the velocity gradients, hence

$$\dot{a}_{\alpha\beta} = v,_{\alpha}.a_{\beta} + a_{\alpha}.v,_{\beta} = 0. \tag{8\_3.2}$$

From equation (5_5.15) putting $Y_{\alpha\beta}= 0$, we write the following solution for (8_3.2)

$$v_{,\alpha} = \Omega \times a_\alpha = \dot{a}_\alpha \tag{8\_3.3}$$

where Ω is the angular velocity vector.

Then from (8_3.2)

$$\left(\Omega \times a_\alpha\right) . a_\beta + \left(\Omega \times a_\beta\right) . a_\alpha = 0. \tag{8\_3.4}$$

Differentiation of equation (8_3.3) with respect to β, gives

$$v_{,\alpha\beta} = \Omega_{,\beta} \times a_\alpha + \Omega \times a_{\alpha,\beta}. \tag{8\_3.5}$$

Interchanging α and β and subtracting, we have

$$v_{,\alpha\beta} - v_{,\beta\alpha} = \Omega_{,\beta} \times a_\alpha + \Omega \times a_{\alpha,\beta} - \Omega_{,\alpha} \times a_\beta - \Omega \times a_{\beta,\alpha}. \tag{8\_3.6}$$

Since $\quad a_{\alpha,\beta} = a_{\beta,\alpha} \quad , \quad v_{,\alpha\beta} - v_{,\beta\alpha} = 0,$

$$\Omega_{,\beta} \times a_\alpha = \Omega_{,\alpha} \times a_\beta. \tag{8\_3.7}$$

Now, by considering β=1 and α=2 (8_3.7) becomes

$$\Omega_{,1} \times a_2 = \Omega_{,2} \times a_1. \tag{8\_3.8}$$

We know that a_1 and a_2 lie in the plane of the surface, therefore (8_3.8) must lie in the direction of the normal and hence, $\Omega_{,\beta}$ must lie in the plane of the surface, then

$$\Omega_{,\beta} . n = 0. \tag{8\_3.9}$$

Multiplying equation (8_3.7) by **n** and using the triple product, we write

$$\left(\overline{\Omega},_{\beta} \times a_{\alpha}\right).n = \left(\overline{\Omega},_{\alpha} \times a_{\beta}\right).n$$

$$\overline{\Omega},_{\beta} .\varepsilon_{\alpha\lambda}a^{\lambda} = \overline{\Omega},_{\alpha} \varepsilon_{\beta\lambda}a^{\lambda}. \qquad (8\_3.10)$$

Setting $\alpha = 1$ and $\beta = 2$, (8_3.10) becomes

$$\overline{\Omega},_{2} .\varepsilon_{1\lambda}a^{\lambda} = \overline{\Omega},_{1} \varepsilon_{2\lambda}a^{\lambda}$$

$$\overline{\Omega},_{2} a^{2} = - \overline{\Omega},_{1} a^{1}$$

and lastly we have

$$\overline{\Omega},_{\beta}.a^{\beta} = 0. \qquad (8\_3.11)$$

If we collect the inextensional deformation conditions together we write

$$\left.\begin{aligned} \overline{\Omega},_{\beta}.n &= 0 \\ \overline{\Omega},_{\beta}.a^{\beta} &= 0 \end{aligned}\right\}. \qquad (8\_3.12)$$

Equations (8_3.12) can directly be obtained from (5_5.23) and (5_5.24) by putting $Y_{\alpha\beta} = 0$. These equations are identical to those obtained in the static equilibrium of unloaded reciprocal surface (8_2.17).

The requirement then for designing a membrane shell is to show that there is no solution to (8_3.12) satisfying the boundary conditions other than $\overline{\Omega} = 0$ (or $\overline{\Omega} =$ constant for a shell in space).

Equations (8_3.12) can be written as in (5_5.13) and (5_5.14) using the relation in (5_5.11)

$$\boldsymbol{\Omega}_{,\beta}\cdot\mathbf{a}^{\beta} = \Omega^{\beta}|_{\beta} - \Omega\, b^{\beta}_{\beta} = 0 \tag{8\_3.13}$$

$$\boldsymbol{\Omega}_{,\alpha}\cdot\mathbf{n} = \Omega^{\beta} b_{\beta\alpha} + \Omega|_{\alpha} = 0. \tag{8\_3.14}$$

Therefore, to write the single partial differential equation for the inextensional deformation we proceed as we did before in (5_6.26) for the general deformation, starting by multiplying (8_3.14) by $\varepsilon^{\xi\eta}\varepsilon^{\rho\alpha}\, b_{\xi\rho}$

$$-\,\Omega|_{\alpha}\, \varepsilon^{\xi\eta}\varepsilon^{\rho\alpha}\, b_{\xi\rho} = \Omega^{\beta}\varepsilon^{\xi\eta}\varepsilon^{\rho\alpha}\, b_{\xi\rho} b_{\beta\alpha}. \tag{8\_3.15}$$

Using (5_6.22), we write

$$-\,\Omega|_{\alpha}\, \varepsilon^{\xi\eta}\varepsilon^{\rho\alpha}\, b_{\xi\rho} = \Omega^{\beta}\, \delta^{\eta}_{\beta}\, K = \Omega^{\eta}\, K \tag{8\_3.16}$$

$$\Omega^{\eta}|_{\lambda} = -\left[\frac{\Omega|_{\alpha}\ \varepsilon^{\xi\eta}\varepsilon^{\rho\alpha}\ b_{\xi\rho}}{K}\right]\Big|_{\lambda}. \tag{8\_3.17}$$

Using the Codazzi equations (3_3.54), (8_3.17) becomes

$$\Omega^{\eta}|_{\eta} = -\,\varepsilon^{\xi\eta}\varepsilon^{\rho\alpha}\, b_{\xi\rho}\left[\frac{\Omega|_{\alpha}}{K}\right]\Big|_{\eta}. \tag{8\_3.18}$$

Substituting (8_3.18) into (8_3.13), we finally write

$$\varepsilon^{\xi\lambda}\varepsilon^{\rho\gamma}\, b_{\xi\rho}\left[\frac{\Omega|_{\gamma}}{K}\right]\Big|_{\lambda} + \Omega\, b^{\lambda}_{\lambda} = 0. \tag{8\_3.19}$$

Equation (8_3.19) can be obtained directly from (5_6.26) by just omitting the rate of membrane strain $Y_{\alpha\beta}$ to end up with a single second order partial differential equation which describes the inextensional modes of deformation. Again, the above equation is applicable provided K (the Gaussian curvature) is different from zero. For developable surfaces instead, a distinct procedure has to be followed and will be shown later.

Equation (8_3.19) when developed is a second order partial different equation in the unknown normal component of angular velocity. It will be shown that its solution depends strongly on the form of the surface of the shell, in particular, on the sign of Gaussian curvature.

8_4 Membrane theory and inextensional deformation relationships

In what follows, we will look at the problem of analogy when the shell surface is not a plane, this will exclude the possibility of the dominance of the bending stresses and give rise to the dominance of the membrane stresses. In surfaces where the membrane stresses are predominant, a membrane theory is expected to be able to cover the analysis (if valid). However, as argued before in the introduction, predominantly in_plane stresses in membrane shells give rise to flexural disturbances at the edges, and pure bending theory treatment of surfaces requires in_plane strains at the edges to prevent the structure from being a mechanism.

Equations (8_3.18) permits the introduction of a new quantity, such that equation (8_3.7) becomes

$$\Omega,_{\alpha} \times \mathbf{a}_{\beta} = c_{\alpha\beta}\mathbf{n}. \tag{8\_4.1}$$

Then,

$$c_{\alpha\beta} = c_{\beta\alpha}. \tag{8\_4.2}$$

Multiplication of both sides of (8_4.1) by **n** yields

$$\Omega,_{\alpha} = c_{\alpha\nu}\varepsilon^{\rho\nu}\mathbf{a}_{\rho}. \tag{8\_4.3}$$

Differentiating (8_4.3) with respect to β, then interchanging α and β and subtracting, with the use of Gauss's equations (3_3.51) we get

$$\varepsilon^{\alpha\beta}\,\Omega,_{\alpha\beta} = \varepsilon^{\alpha\beta}\varepsilon^{\rho\nu}c_{\alpha\nu}|_{\beta}\mathbf{a}_{\rho} + \varepsilon^{\alpha\beta}\varepsilon^{\rho\nu}c_{\alpha\nu}b_{\beta\rho}\mathbf{a}_{3} = 0 \tag{8\_4.4}$$

since $\Omega,_{\alpha\beta}$ is symmetric.

The first term in the right hand side of (8_4.4) is equal to zero since $c_{\alpha\nu}$ is also symmetric, i.e

$$c_{\alpha\nu}|_{\beta} = c_{\beta\nu}|_{\alpha} \tag{8\_4.5}$$

then,

$$\varepsilon^{\alpha\beta}\varepsilon^{\rho\nu}c_{\alpha\nu}b_{\beta\rho} = 0 \tag{8\_4.6}$$

Equations (8_4.2), (8_4.5) and (8_4.6) are the bending strain relations in the case of flexure. They can also be obtained directly from the general equations of compatibility (5_6.20) by, using (6_2.8) and (6_3.2) and, assuming $Y_{\alpha\beta} = 0$. Collected together, we write

$$\left.\begin{aligned} c_{\alpha\nu} &= c_{\nu\alpha} \\ \varepsilon^{\alpha\beta}\varepsilon^{\nu\rho}c_{\alpha\nu}b_{\beta\rho} &= 0 \\ \left[\varepsilon^{\alpha\beta}\ \varepsilon^{\nu\rho}\ c_{\alpha\nu}\right]|_{\beta} &= 0 \end{aligned}\right\}. \tag{8_4.7}$$

Now, by introducing a new quantities $Q^{\beta\lambda}$ defined as

$$Q^{\beta\lambda} = \varepsilon^{\beta\alpha}\varepsilon^{\lambda\nu}c_{\alpha\nu} \tag{8_4.8}$$

then, the equations in (8_4.7) become

$$\left.\begin{aligned} Q^{\beta\lambda} &= Q^{\lambda\beta} \\ Q^{\beta\lambda}|_{\beta} &= 0 \\ Q^{\beta\lambda}b_{\beta\lambda} &= 0 \end{aligned}\right\}. \tag{8_4.9}$$

The above three equations are equivalent to the equilibrium equations of the membrane surface when the external loads are set equal to zero, see (8_2.4), in which the tensor $Q^{\beta\lambda}$ is equivalent to the stress tensor $n^{\beta\lambda}$. Then, equation (8_4.8) represents the analogy between the static and geometric equations in the equilibrium and compatibility relations and shows the one to one correspondence between them.

$$n^{\beta\lambda} \longrightarrow Q^{\beta\lambda}. \tag{8_4.10}$$

The above equivalence means that, knowing for instance the all non_trivial flexures of a given surface we can derive also the corresponding non_trivial states of stress in the unloaded membrane shell.

The requirement to prevent inextensional deformation (i.e. small flexure) is to show that there is no solution to the compatibility equations (8_4.7) other than setting $\varepsilon^{\alpha\beta}\varepsilon^{\nu\rho}c_{\alpha\nu} = 0$.
In terms of stresses, consequently, we must show the existence of a solution to the complete equations of the membrane theory.

The most important feature of the above conditions is that the boundary conditions necessary for preventing the inextensional deformation are those which are necessary for the equilibrium of the membrane shell. Thus, designing a membrane shell means eliminating small flexural bending which also means preventing the structure from behaving like a mechanism.

If we, take the vector product of equation (5_5.11) by the normal and, apply the vector product rules, we write

$$\bar{\Omega} \times \mathbf{n} = \left[\Omega^{\beta}\mathbf{a}_{\beta} + \Omega\,\mathbf{n}\right] \times \mathbf{n}$$

$$= \Omega^{\beta}\left[\mathbf{a}_{\beta} \times \mathbf{n}\right] = \Omega^{\beta}\varepsilon_{\rho\beta}\mathbf{a}^{\rho}. \qquad (8\_4.11)$$

Comparison of (8_4.11) with (5_5.16), shows that

$$\bar{\mathbf{n}} = \bar{\Omega} \times \mathbf{n}. \qquad (8\_4.12)$$

Differentiating with respect to α, we write

$$\bar{\mathbf{n}}_{,\alpha} = \bar{\Omega}_{,\alpha} \times \mathbf{n} + \bar{\Omega} \times \mathbf{n}_{,\alpha}. \qquad (8\_4.13)$$

From the rate of change of the second fundamental form of the surface we have

$$b^{\cdot}_{\alpha\beta} = -\,\dot{n}_{,\alpha}\,.\,a_{\beta} - n_{,\alpha}\,.\,\dot{a}_{\beta}.$$

Using (8_4.13) and (8_3.3), the above equation becomes

$$b^{\cdot}_{\alpha\beta} = -\left[\Omega_{,\alpha} \times n + \Omega \times n_{,\alpha}\right].a_{\beta} - n_{,\alpha}\,.\left[\Omega \times a_{\beta}\right]$$

$$= -\left[\Omega_{,\alpha} \times n\right].a_{\beta}. \qquad (8\_4.14)$$

Comparison of (8_4.14) with (8_4.1) shows that

$$b^{\cdot}_{\alpha\beta} = c_{\alpha\beta}. \qquad (8\_4.15)$$

The equality in (8_4.15) shows that the equations in (8_4.5) and (8_4.6) are statements of the fact that Codazzi equations and Gauss's theorem of the surface continue to hold as the surface deforms.

Before closing this section, let us introduce two crucial quantities which are the rates of change of the normal curvature and twist, $k^{\cdot}_{n}$ and $\tau^{\cdot}$. These two expressions will be used in the discussion of the boundary conditions of inextensional shell structure deformations. The rate of change of the normal curvature in the inextensional modes of deformation will be obtained by differentiating the first equation of (3_3.31) with respect to time, and assuming that the rate of membrane strains are all equal to zero, i.e $Y_{\alpha\beta} = 0$.

$$\dot{k}_n = -\frac{\partial}{\partial t}\left[\frac{b_{\alpha\beta}\, d\vartheta^{\alpha} d\vartheta^{\beta}}{a_{\gamma\eta}\, d\vartheta^{\gamma} d\vartheta^{\eta}}\right] = -\frac{\partial}{\partial t}\left[\frac{b_{\alpha}^{\lambda}\, a_{\beta\lambda} d\vartheta^{\alpha} d\vartheta^{\beta}}{a_{\gamma\eta}\, d\vartheta^{\gamma} d\vartheta^{\eta}}\right].$$

With,

$$Y_{\alpha\beta} = 1/2\ \bar{a}_{\alpha\beta} = 0, \tag{8_4.16}$$

$$\dot{k}_n = -\frac{\bar{b}_{\alpha}^{\lambda}\, a_{\beta\lambda} d\vartheta^{\alpha} d\vartheta^{\beta}}{a_{\gamma\eta}\, d\vartheta^{\gamma} d\vartheta^{\eta}}. \tag{8_4.17}$$

Substituting (8_4.16) into (5_6.8), for inextensional deformation, we write

$$\bar{b}_{\lambda}^{\alpha} = \varepsilon_{\rho\lambda}\, \varepsilon_{\gamma\beta}\, ß^{\gamma\rho}\, a^{\alpha\beta} \tag{8_4.18}$$

and (8_4.17), becomes

$$\dot{k}_n = -\frac{\varepsilon_{\rho\alpha}\, \varepsilon_{\gamma\beta}\, ß^{\gamma\rho} d\vartheta^{\alpha} d\vartheta^{\beta}}{a_{\gamma\eta}\, d\vartheta^{\gamma} d\vartheta^{\eta}}. \tag{8_4.19}$$

From the second equation of (3_3.31), the rate of change of the twist in the inextensional modes of deformation is

$$\dot{\tau} = \frac{\partial}{\partial t}\left[\frac{b_{\alpha}^{\lambda}\, \varepsilon_{\beta\lambda} d\vartheta^{\alpha} d\vartheta^{\beta}}{a_{\gamma\eta}\, d\vartheta^{\gamma} d\vartheta^{\eta}}\right] \tag{8_4.20}$$

and after differentiation and substitution of (8_4.16), we write

$$\tau^{*} = \frac{\varepsilon_{\eta\beta}\,\varepsilon_{\gamma\upsilon}\,\varepsilon_{\alpha\lambda}a^{\lambda\upsilon}\,\,{}_{B}{}^{\gamma\eta}d\vartheta^{\alpha}d\vartheta^{\beta}}{a_{\gamma\eta}\,\,d\vartheta^{\gamma}d\vartheta^{\eta}}. \tag{8_4.21}$$

8_4.1 The scalar ψ

Any vector which is a function of the two coordinates ϑ^{α} can be considered to represent a surface in a suitable space. Thus $\mathbf{r}(\vartheta^1,\vartheta^2)$ is the reference surface for a shell and its velocity $\mathbf{v}(\vartheta^1,\vartheta^2)$ and angular velocity $\boldsymbol{\Omega}(\vartheta^1,\vartheta^2)$ are surfaces in velocity and angular velocity spaces.

In addition to the three surfaces $\mathbf{r}$,$\mathbf{v}$ and $\boldsymbol{\Omega}$, we can write the new $\mathbf{u}$ given by

$$\mathbf{u}_{,\alpha} = \mathbf{r} \times \boldsymbol{\Omega}_{,\alpha}. \tag{8_4.22}$$

This satisfies $\mathbf{u}_{,\alpha\beta} = \mathbf{u}_{,\beta\alpha}$ since

$$\mathbf{u}_{,\alpha\beta} = \mathbf{r}_{,\beta} \times \boldsymbol{\Omega}_{,\alpha} + \mathbf{r} \times \boldsymbol{\Omega}_{,\alpha\beta} = \mathbf{u}_{,\beta\alpha}$$

from (8_3.7).

Therefore the difference of (8_3.3) and (8_4.22) gives

$$\begin{aligned} \mathbf{v}_{,\alpha} - \mathbf{u}_{,\alpha} &= \boldsymbol{\Omega} \times \mathbf{a}_{\alpha} - \mathbf{r} \times \boldsymbol{\Omega}_{,\alpha} \\ &= \boldsymbol{\Omega} \times \mathbf{a}_{\alpha} + \boldsymbol{\Omega}_{,\alpha} \times \mathbf{r} = \left[\boldsymbol{\Omega} \times \mathbf{r} \right]_{,\alpha} \end{aligned} \tag{8_4.23}$$

and hence,

$$\mathbf{v} - \mathbf{u} = \left[\overline{\Omega} \times \mathbf{r} \right]. \tag{8_4.24}$$

Equation (8_4.22) shows that $\mathbf{u},_{\alpha}$ is perpendicular to $\mathbf{r}$, then

$$\mathbf{u},_{\alpha} \cdot \mathbf{r} = 0 \tag{8_4.25}$$

and then

$$\left[\mathbf{u} \cdot \mathbf{r} \right],_{\alpha} = \mathbf{u} \cdot \mathbf{a}_{\alpha}. \tag{8_4.26}$$

Let the scalar ψ be

$$\psi = \mathbf{u} \cdot \mathbf{r}. \tag{8_4.27}$$

Therefore, using (8_4.26)

$$\psi,_{\alpha} = \mathbf{u} \cdot \mathbf{r},_{\alpha}. \tag{8_4.28}$$

Equations (8_4.27) and (8_4.28) define $\mathbf{u}$ uniquely as

$$\mathbf{u} = \psi,_{\alpha} \mathbf{a}^{\alpha} + \frac{\left[\psi - \psi,_{\alpha} \mathbf{a}^{\alpha} \cdot \mathbf{r} \right]}{\mathbf{n} \cdot \mathbf{r}} \mathbf{n}. \tag{8_4.29}$$

Vector multiplication of (8_4.22) by $\mathbf{n}$ yields

$$\mathbf{u},_{\alpha} \times \mathbf{n} = \left[\mathbf{r} \times \overline{\Omega},_{\alpha} \right] \times \mathbf{n}. \tag{8_4.30}$$

Using the vector triple product (8_4.30) becomes

$$\mathbf{u},_{\alpha} \times \mathbf{n} = \overline{\Omega},_{\alpha} \left[\mathbf{r} \cdot \mathbf{n} \right] - \mathbf{r} \left[\overline{\Omega},_{\alpha} \cdot \mathbf{n} \right] \tag{8_4.31}$$

with the help of (8_3.9)

$$\boldsymbol{\Omega},_{\alpha} = \frac{\mathbf{u},_{\alpha} \times \mathbf{n}}{\mathbf{r}\,.\,\mathbf{n}}. \qquad (8\_4.32)$$

As $\boldsymbol{\Omega},_{\alpha\beta} = \boldsymbol{\Omega},_{\beta\alpha}$, differentiating (8_4.32) with respect to β, interchanging α and β then subtracting, we write

$$\varepsilon^{\alpha\beta}\, \mathbf{u},_{\alpha} \times \left[\frac{\mathbf{n}}{\mathbf{r}\,.\,\mathbf{n}}\right],_{\beta} = 0. \qquad (8\_4.33)$$

We know that $\mathbf{u},_{\alpha}$ is perpendicular to $\mathbf{r}$ and we also have

$$\left[\frac{\mathbf{n}}{\mathbf{r}\,.\,\mathbf{n}}\right],_{\beta} .\mathbf{r} = \left[\frac{\mathbf{r}\,.\,\mathbf{n}}{\mathbf{r}\,.\,\mathbf{n}}\right],_{\beta} - \left[\frac{\mathbf{n}}{\mathbf{r}\,.\,\mathbf{n}}\right] .\mathbf{r},_{\beta} = 0.$$

Thus $\left[\frac{\mathbf{n}}{\mathbf{r}\,.\,\mathbf{n}}\right],_{\beta}$ is also perpendicular to $\mathbf{r}$ and scalar multiplying (8_4.33) by $\mathbf{n}$, gives

$$\varepsilon^{\alpha\beta}\, \mathbf{u},_{\alpha} \cdot \left\{\left[\frac{\mathbf{n}}{\mathbf{r}\,.\,\mathbf{n}}\right],_{\beta} \times \mathbf{n}\right\} = -\,\varepsilon^{\alpha\beta}\, \mathbf{u},_{\alpha}\left\{\frac{b_{\beta\rho}\mathbf{a}^{\rho} \times \mathbf{n}}{\mathbf{r}\,.\,\mathbf{n}}\right\}$$

$$= -\,\frac{\varepsilon^{\alpha\beta}\varepsilon^{\upsilon\rho} b_{\beta\rho}\, \mathbf{u},_{\alpha} \cdot \mathbf{a}_{\upsilon}}{\mathbf{r}\,.\,\mathbf{n}} = 0. \qquad (8\_4.34)$$

From (8_4.29), we have

$$\mathbf{u},_{\lambda}.\mathbf{a}_{\alpha} = \psi,_{\alpha\lambda} + \psi,_{\rho}\, \mathbf{a}^{\rho},_{\lambda}.\mathbf{a}_{\alpha} + \frac{\left[\psi - \psi,_{\gamma}\, \mathbf{a}^{\gamma} .\mathbf{r}\right]}{\mathbf{n}\,.\,\mathbf{r}}\, \mathbf{n},_{\lambda}.\mathbf{a}_{\alpha}$$

$$= \psi,_{\alpha\lambda} - \psi,_{\rho}\, \Gamma^{\rho}_{\lambda\alpha} + \frac{\left[\psi - \psi,_{\gamma}\, \mathbf{a}^{\gamma} .\mathbf{r}\right]}{\mathbf{n}\,.\,\mathbf{r}}\, \mathbf{n},_{\lambda}.\mathbf{a}_{\alpha}$$

$$= \psi|_{\alpha\lambda} - b_{\alpha\lambda}\, \frac{\left[\psi - \psi,_{\gamma}\, \mathbf{a}^{\gamma} .\mathbf{r}\right]}{\mathbf{n}\,.\,\mathbf{r}}. \qquad (8\_4.35)$$

Substituting (8_4.35) into (8_4.34) we get one equation in the one unknown ψ

$$\varepsilon^{\alpha\beta}\varepsilon^{\upsilon\rho}b_{\beta\rho}\left[\psi|_{\upsilon\alpha} - b_{\upsilon\alpha}\frac{\left[\psi - \psi,_{\gamma}\, a^{\gamma} .r\right]}{n\ .\ r}\right] = 0$$

$$= \varepsilon^{\alpha\beta}\varepsilon^{\upsilon\rho}b_{\beta\rho}\psi|_{\upsilon\alpha} - 2\ K\ \frac{\left[\psi - \psi,_{\gamma}\, a^{\gamma} .r\right]}{n\ .\ r} = 0. \qquad (8\_4.36)$$

Substituting (8_4.35) into (8_4.36) we take

$$\Omega,_{\alpha} = \left[\psi|_{\lambda\alpha} - b_{\lambda\alpha}\frac{\left[\psi - \psi,_{\gamma}\, a^{\gamma} .r\right]}{n\ .\ r}\right]\frac{\varepsilon^{\rho\lambda}a_{\rho}}{r\ .\ n}. \qquad (8\_4.37)$$

Lastly

$$c_{\alpha\beta} = \Omega,_{\alpha} .\left[a_{\beta} \times n\right] = \Omega,_{\alpha} .\varepsilon_{\rho\beta}a^{\rho}$$

$$= \frac{\left[\psi|_{\beta\alpha} - b_{\beta\alpha}\frac{\left[\psi - \psi,_{\gamma}\, a^{\gamma} .r\right]}{n\ .\ r}\right]}{n\ .\ r}. \qquad (8\_4.38)$$

8_5 Conclusion

The inextensional deformation of a surface must satisfy certain differential equations containing velocities, angular velocities etc. The aim is to reduce these equations to one equation and then find the boundary conditions necessary to prevent inextensional deformation.

Equation (8_3.19) contains only the unknown Ω, the normal component of the angular velocity vector $\overline{\Omega}$ and equation (8_4.36) contain only ψ. There are other ways of producing a single equation, for example one equation can be produced for the component of velocity in a fixed direction (see plane coordinates in Green & Zerna (1968)).

CHAPTER NINE

THE RIGIDITY OF MEMBRANE SHELLS

9_1 Introduction

In the previous chapter, in the membrane theory of equilibrium as well as the inextensional theory of deformation, it has been concluded that, if one wants to design a shell that will work primarily as a membrane shell, one should avoid designing a structure that behaves as a mechanism. The mechanism in a structure can be seen from two different points of view, from the static and from the kinematic points of view.

A membrane shell may be statically determinate, statically indeterminate or a mechanism. In the static equilibrium of membrane shell, the structure is not a mechanism if one can find solutions to the equations which satisfy the boundary conditions for an arbitrary load. The satisfaction of boundary conditions, as stated before in section (8_2.3), is a result of satisfying the external load and then the homogeneous problem of membrane theory.
The structure is statically determinate if the solutions are unique otherwise it is statically indeterminate.

From the point of view of kinematic of displacement and rotation, it was found that the structure would behave as a mechanism if the inextensional deformation is not prevented properly. An elucidatory example, is the pin_ bar chain in fig.(9_1.1), where the strains are proportional to the square of the transverse displacement, Kuznetsov (1989).

$$l^2 + Y^2 = \left[\, l(1+\varepsilon) \,\right]^2 \longrightarrow Y \simeq l\sqrt{2\varepsilon}.$$

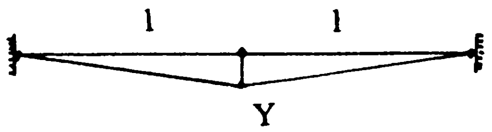


fig.(9_1.1) Infinitesimal displacement (mechanism)

Inextensional deformation is not possible if it can be shown that there is no distribution of bending strains which produces inextensional deformation and satisfies at the same time the boundary conditions

The bending strains $c_{\alpha\nu}$ produce inextensional deformation if (8_4.7) are satisfied and therefore the boundary conditions have to be such that the only solution to (8_4.7) is $c_{\alpha\nu} = 0$.

The character of the equilibrium equations of the membrane theory and also of those of inextensional deformations were found to be mathematically dependent of the geometry of the surface in question. Their solutions will be subject to some fixed boundary conditions. This would link the aspect of rigidity of structures to the geometry of the surface and also to the type of boundary support adopted.

In what follows, it is aimed to try to isolate the above two factors that are responsible for the rigidity of the structure. If we consider closed shell surfaces, then the concept of boundary conditions loses its meaning, and we end up with a shell surface defined only by its geometry.

Calladine (1983) stated that "Any one who has built children's toys from thick paper or thin card will be familiar with the striking fact that a closed box is rigid, while an open box is easily deformable". The rigidity aimed for in the above statement is relative to open flexible surfaces, and the question that it is aimed to answer here, is absolute and concerned with, whether any closed surface of any given Gaussian curvature is rigid?

Before trying to answer the question, let us define a number of terms which will be repeatedly used.
Coxeter (1961) has stated that "....In particular, an isometry (or *congruent transformation*) is a transformation which preserves length, so that if (P,P') and (Q,Q') are two pairs of corresponding points, we have PQ = P'Q' :PQ and P'Q'are congruent segments. For instance a *rotation* of the plane about P (or about a line through P perpendicular to the plane) is an isometry having P as an invariant point, but a *translation* (or "parallel displacement") has no invariant point :every point is moved.
A reflection is the special kind of isometry in which the invariant points consists of all points on line (or a plane)

called the *mirror*.

A still simpler kind of transformation (so simple that it may at first seem too trivial to be worth mentioning) is the *identity*, which leaves every point unchanged. The result of applying several transformations successively is called their *product*. If the product of two transformations is the identity, each called the *inverse* of the other, and their product in the reverse order is again the identity."

The rigidity from the point of view of pure mathematics is related to whether an embedded closed surface in R^3 after a given motion remains globally isometric to the initial surface. The isometry in this case on top of being local, is also concerned with the second fundamental form of the surface being not changeable in absolute value after the motion. This confines the motion to an Euclidean motion. Therefore a rigid surface is a surface which is not bendable. A bendable surface is a surface which can undergo a continuous deformation without stretching its middle surface.

In addition to these properties, Spivak (1979) defined also some other surfaces which are isometric to each other at least in two states, and called them warpable surfaces. A bendable surface, he stated, is obviously warpable but it is not a priori clear whether there are any warpable surfaces which are not bendable.

A warpable surface is a surface which has equal line elements to a corresponding surface, but obtained without continuous bending. This seems as if the passage from one state to another is achieved by cutting and then gluing the surface again. These surfaces, apparently, are obtained through buckling under large deformation, in which the folding constitute the general aspect of deformation. In this context Hoff, Tsai_Chen Soong & Sendelbeck (1969) have stated that " ..there exist polyhedral surfaces whose line elements have the same length as the corresponding line elements of the original middle surface of the circular cylindrical shell.....The circular cylinder can be transformed into this polyhedral surface through an inextensional deformation; by this statement it is not implied, however, that a continuous inextensional path leads from the initial shape to the final shape. On the contrary, the transition takes place when the shell suddenly snaps into its final configuration from an intermediate state in which both extensional and bending deformation are present.", fig.(9_1.2).

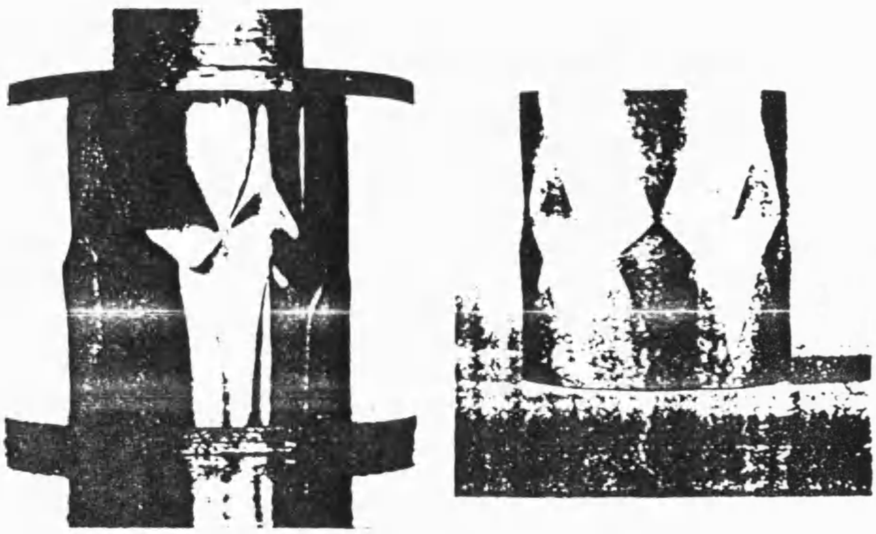


fig.(9_1.2) buckled cylinder, after Hoff, et al. (1969)

Thus, if a surface is warpable but not bendable, this does not concern the membrane theory, however, it is one of the aspects of buckling of shells.

A surface which is not bendable is termed rigid. An infinitesimally rigid surface is one which requires extensional strains which are proportional to the magnitude of the displacement. However, a surface which only requires membrane strains proportional to the displacement raised to a power greater than one is rigid. A rigid surface might not be infinitesimally rigid, whereas an infinitesimally rigid surface must be rigid.

An infinitesimally bendable surface is a surface which can undergo a small continuous inextensional deformation. A bendable surface is one which can undergo continuous finite inextensional deformation.

In order for a membrane shell to work under arbitrary load it must be infinitesimally rigid. However, a prestressed surface will not be a mechanism if it is rigid, but not necessarily infinitely rigid. for example a prestressed two_way cable net is rigid, but not infinitesimally rigid.

The following table shows the different relationships between the different surface's properties with respect to the rigidity.

<table>
<tr><th></th><th></th><th colspan="6">Given Property</th></tr>
<tr><th></th><th></th><th>Warpable</th><th>Bendable</th><th>Infini. Bend.</th><th>Rigid</th><th>Infini Rigid</th><th>Unwarp_ble</th></tr>
<tr><td rowspan="6">Does this property apply</td><td>Warpable Finite chg</td><td>✓</td><td>✓</td><td>?</td><td>?</td><td>?</td><td>X</td></tr>
<tr><td>Bendable</td><td>?</td><td>✓</td><td>?</td><td>X</td><td>X</td><td>X</td></tr>
<tr><td>Infinite. Bendable</td><td>?</td><td>✓</td><td>✓</td><td>?</td><td>X</td><td>?</td></tr>
<tr><td>Rigid</td><td>?</td><td>X</td><td>?</td><td>✓</td><td>✓</td><td>✓</td></tr>
<tr><td>Infinite. Rigid</td><td>?</td><td>X</td><td>X</td><td>?</td><td>✓</td><td>?</td></tr>
<tr><td>Unwarpable</td><td>X</td><td>X</td><td>?</td><td>?</td><td>?</td><td>✓</td></tr>
</table>

✓ = yes

X = no

? = possibly

Table (2): Surface properties Relationships

9_2 Cohn_vossen's theorem on the rigidity of ovaloids

An ovaloid is a complete compact, strictly convex surface. The strict convexity of the surface, according to Buchin (1982), means that the Gaussian curvature is positive everywhere on the surface. Examples of these surfaces include the sphere, and more particularly an egg shell from which the name ovaloid follows.

According to Chern (1967), the rigidity theorem of Cohn_vossen can be stated as follows:
"An isometry between two closed convex surfaces is established either by a motion or by a motion and a reflection.".
The proof of this theorem has been supplied by many geometers among whom we mention Chern (1967), Klingenberg (1978), Spivak (1979) and Buchin (1982).

According to Klingenberg (1978) there exists an isometry if and only if there exists a diffeomorphism (differential mapping) which preserves the first fundamental form of the surface and preserves the second fundamental form up to sign (i.e. $a_{\alpha\beta} = \overset{*}{a}_{\alpha\beta}$, $b_{\alpha\beta} = \pm \overset{*}{b}_{\alpha\beta}$).

9_2.1 Strictly closed convex surfaces with (K>0)

The following proof is based on the work of G. Herglotz given by Chern (1967).
To prove the theorem in section (9_2), let us consider a strictly

closed convex surface S in an Euclidean space E^3 on which a point $P(\vartheta^1, \vartheta^2)$ is given by the position vector $\mathbf{r}(\vartheta^1, \vartheta^2)$. The coordinates ϑ^1, ϑ^2 are the parametric curves on the surface. The position vector is supposed to be twice continuously differentiable, and the base vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are everywhere linearly independent. Let $\mathbf{n}$ be the unit normal vector, so S is orientable. The first and second fundamental forms of the surface are from (3_3.15) and (3_3.25) and are respectively

$$\begin{aligned} dr.dr &= a_{\alpha\beta} d\vartheta^{\alpha} d\vartheta^{\beta} \\ dr.dn &= -\ b_{\alpha\beta} d\vartheta^{\alpha} d\vartheta^{\beta}. \end{aligned} \qquad (9\_2.1)$$

As before the symbols H and K denote respectively the mean and Gaussian curvatures. Let the second surface be S^* where its surface quantities will be given the same symbols with asterisks.

If we choose the local coordinates such that corresponding points on S and S^* have the same local coordinates , then under isometry the metric surface tensors are equal (ie $a_{\alpha\beta} = a^*_{\alpha\beta}$) and the same is true for the Christoffel symbols. From (3_3.38) and (3_3.39), the mean and Gaussian curvatures for both surfaces are

$$\begin{aligned} H &= (1/\ 2a) \left\{ a_{22} b_{11} - 2b_{12} a_{12} + a_{11} b_{22} \right\} \\ H^* &= (1/\ 2a) \left\{ a_{22} b^*_{11} - 2b^*_{12} a_{12} + a_{11} b^*_{22} \right\} \end{aligned} \qquad (9\_2.2)$$

where $$a^* = a = a_{11} a_{22} - (a_{12})^2 \qquad (9\_2.3)$$

and

$$K = \frac{|\, b_{\alpha\beta} \,|}{a} = \frac{b_{11}b_{22}-(b_{12})^2}{a}$$

$$K^* = \frac{|\, b^*_{\alpha\beta} \,|}{a^*} = \frac{b^*_{11}b^*_{22}-(b^*_{12})^2}{a^*}. \tag{9_2.4}$$

Since S is isometric to S^*, then by Gauss's theorem we have

$$K = K^* \text{ and } a = a^* \tag{9_2.5}$$

so that by (9_2.4)

$$|b_{\alpha\beta}| = |b^*_{\alpha\beta}| = K\ a = K^*\ a^*. \tag{9_2.6}$$

We now have to show that

$$b_{\alpha\beta} = b^*_{\alpha\beta}. \tag{9_2.7}$$

We define a new quantity

$$j = 2\ K - \begin{Bmatrix} \left(1/\sqrt{a}\right)\left\{b^*_{11}-b_{11}\right\} & \left(1/\sqrt{a}\right)\left\{b^*_{12}-b_{12}\right\} \\ \left(1/\sqrt{a}\right)\left\{b^*_{12}-b_{12}\right\} & \left(1/\sqrt{a}\right)\left\{b^*_{22}-b_{22}\right\} \end{Bmatrix}$$

$$j = \left(1/\ a\right)\ \left\{ b_{11}b^*_{22} - 2b_{12}b^*_{12} + b_{22}b^*_{11} \right\}. \tag{9_2.8}$$

Now, from (3_3.54) the equations of Codazzi for S^* are

$$\varepsilon^{\alpha\gamma}\, b^{*}_{\alpha\beta}|_{\gamma} = 0$$

which, if developed becomes

$$b^{*}_{11,2} - b^{*}_{12,1} - b^{*}_{11}\Gamma^{1}_{12} - b^{*}_{12}\left(\Gamma^{2}_{12} - \Gamma^{1}_{11} \right) + b^{*}_{22}\Gamma^{2}_{11} = 0$$

$$b^{*}_{12,2} - b^{*}_{22,1} - b^{*}_{11}\Gamma^{1}_{22} - b^{*}_{12}\left(\Gamma^{2}_{22} - \Gamma^{1}_{12} \right) + b^{*}_{22}\Gamma^{2}_{12} = 0.$$

Dividing by $\sqrt{a}$, we write

$$\left[\frac{b^{*}_{11}}{\sqrt{a}}\right]_{,2} - \left[\frac{b^{*}_{12}}{\sqrt{a}}\right]_{,1} + \frac{b^{*}_{11}}{\sqrt{a}}\,\Gamma^{2}_{22} - 2\,\frac{b^{*}_{12}}{\sqrt{a}}\,\Gamma^{2}_{12} + \frac{b^{*}_{22}}{\sqrt{a}}\,\Gamma^{2}_{11} = 0$$

$$\left[\frac{b^{*}_{12}}{\sqrt{a}}\right]_{,2} - \left[\frac{b^{*}_{22}}{\sqrt{a}}\right]_{,1} - \frac{b^{*}_{11}}{\sqrt{a}}\,\Gamma^{1}_{22} + 2\,\frac{b^{*}_{12}}{\sqrt{a}}\,\Gamma^{1}_{12} + \frac{b^{*}_{22}}{\sqrt{a}}\,\Gamma^{1}_{11} = 0 \qquad (9\_2.9)$$

Also from (3_3.51), The Gauss equations are

$$\mathbf{a}_{1,1} - \Gamma^{1}_{11}\mathbf{a}_{1} - \Gamma^{2}_{11}\mathbf{a}_{2} - \sqrt{a}\; b_{11}\mathbf{n} = 0$$

$$\mathbf{a}_{1,2} - \Gamma^{1}_{12}\mathbf{a}_{1} - \Gamma^{2}_{12}\mathbf{a}_{2} - \sqrt{a}\; b_{12}\mathbf{n} = 0 \qquad (9\_2.10)$$

$$\mathbf{a}_{2,2} - \Gamma^{1}_{22}\mathbf{a}_{1} - \Gamma^{2}_{22}\mathbf{a}_{2} - \sqrt{a}\; b_{22}\mathbf{n} = 0.$$

Multiplying the above five formulae (9_2.9) and (9_2.10) respectively by $\mathbf{a}_2$, $-\mathbf{a}_1$, $\frac{b^{*}_{22}}{\sqrt{a}}$, $-2\,\frac{b^{*}_{12}}{\sqrt{a}}$ and $\frac{b^{*}_{11}}{\sqrt{a}}$, and adding them, we establish the following

$$\sqrt{a}\ j\ \mathbf{n} = \left[\left\{ \frac{b^*_{22}\ \mathbf{a}_1 - b^*_{12}\ \mathbf{a}_2}{\sqrt{a}} \right\}_{,1} - \left\{ \frac{b^*_{12}\ \mathbf{a}_1 - b^*_{11}\ \mathbf{a}_2}{\sqrt{a}} \right\}_{,2} \right]. \qquad (9\_2.11)$$

We now write

$$p = \mathbf{r} \cdot \mathbf{n} \qquad y_1 = \mathbf{r} \cdot \mathbf{a}_1 \qquad y_2 = \mathbf{r} \cdot \mathbf{a}_2 \qquad (9\_2.12)$$

where $p(\vartheta^1\ \vartheta^2)$ is the oriented distance from the origin to the tangent plane at $\mathbf{r}(\vartheta^1, \vartheta^2)$ and is called the support function. Taking the scalar product of equation (9_2.11) by $\mathbf{r}$, we end up with

$$\sqrt{a}\ j\ p = -\left(\frac{b^*_{22}\ a_{11} - 2b^*_{12}\ a_{12} + b^*_{11}\ a_{22}}{\sqrt{a}} \right) + \left(\frac{b^*_{22}\ y_1}{\sqrt{a}} - \frac{b^*_{12}\ y_2}{\sqrt{a}} \right)_{,1}$$

$$- \left(\frac{b^*_{12}\ y_1}{\sqrt{a}} - \frac{b^*_{11}\ y_2}{\sqrt{a}} \right)_{,2}. \qquad (9\_2.13)$$

If C is a closed curve on the surface S, it divides it into two domains D_1 and D_2. These two domains are supposed to be oriented, therefore C appears as a boundary in opposite senses. By applying Green's theorem for the two domains we get

$$\iint_{D1} j\ p\ dA = \iint_{D1} \left(- \frac{b^*_{22}\ a_{11} + 2b^*_{12}\ a_{12} - b^*_{11}\ a_{22}}{\sqrt{a}} \right) d\vartheta^1\ d\vartheta^2 +$$

$$+\int_{C}\left(\frac{b^{*}_{12}\,y_1}{\sqrt{a}}-\frac{b^{*}_{11}\,y_2}{\sqrt{a}}\right)d\vartheta^1+\left(\frac{b^{*}_{22}\,y_1}{\sqrt{a}}-\frac{b^{*}_{12}\,y_2}{\sqrt{a}}\right)d\vartheta^2.$$

A similar value can be obtained for the domain D_2. Adding the two domains we find the following integral on the surface

$$\iint_{S} j.p\ dA = \iint_{S}\left(-\frac{b^{*}_{22}\,a_{11}+2b^{*}_{12}\,a_{12}-b^{*}_{11}\,a_{22}}{\sqrt{a}}\right)d\vartheta^1 d\vartheta^2 \qquad (9\_2.14)$$

Comparison of the right hand side of equation (9_2.14) and equation (9_2.2) yields

$$\iint_{S} j\ p\ dA = -\,2\iint_{S} H^{*} dA. \qquad (9\_2.15)$$

In particular when S and S^{*} are identical $j = 2K$ and $H^{*} = H$, and then

$$2\iint_{S} K\ p\ dA = -\,2\iint_{S} H\ dA. \qquad (9\_2.16)$$

Subtracting (9_2.16) from (9_2.15) we get

$$\iint_{S}\left(2\,K - j\right)p\ dA = 2\left(\iint_{S} H^{*} dA - \iint_{S} H\ dA\right) \qquad (9\_2.17)$$

which represents Herglotz' integral formula

To discuss equation (9_2.17) we avoid the usual mathematical reasoning based on the discussion of quadratures and allow for the geometrical quantities of the structure to be considered. The left hand side of (9_2.17), is

$$T = 2\,K - j$$

$$= 2\,K - \left(1/\ a\right)\left\{ b_{11}b^{*}_{22} - 2b_{12}b^{*}_{12} + b_{22}b^{*}_{11} \right\}$$

$$= \left(1/\ a\right)\left(2\,K\,a - b_{11}b^{*}_{22} - b_{22}b^{*}_{11} + 2\,b_{12}b^{*}_{12}\right)$$

$$T = \frac{1}{a}\left(2\,K\,a - \frac{b_{11}}{b^{*}_{11}}\left(Ka + [b^{*}_{12}]^{2}\right) - \frac{b^{*}_{11}}{b_{11}}\left(Ka + [b_{12}]^{2}\right) + 2\,b_{12}b^{*}_{12}\right)$$

$$T = \frac{1}{a\,b_{11}b^{*}_{11}}\left(2Kab_{11}b^{*}_{11} - b^{2}_{11}\left(Ka + b^{*\,2}_{12}\right) - b^{*2}_{11}\left(Ka + b^{2}_{12}\right) + 2b_{12}b^{*}_{12}b_{11}b^{*}_{11}\right)$$

$$= -\,\frac{1}{a\,b_{11}b^{*}_{11}}\left(Ka\,\left(b_{11} - b^{*}_{11}\right)^{2} + \left(b_{11}b^{*}_{12} - b^{*}_{11}b_{12}\right)^{2}\right) \qquad (9\_2.18)$$

Dividing by b_{11} and b^{*}_{11} in (9_2.18) is possible since they are both different from zero by $K > 0$, (i.e. there are no asymptotic directions on the surface). By choice of orientation b_{11} and b^{*}_{11} must have the same sign, and since $K \geq 0$ then,

$$T \leq 0. \qquad (9\_2.19)$$

Equation (9_2.18) is equal zero only if

$$\left(b_{11} - b^*_{11}\right) = 0\ ,\ \left(b_{11}b^*_{12} - b^*_{11}b_{12}\right) = 0.$$

Since $K = K^*$ by (9_2.5), then $T = 0$ only if

$$b_{11} = b^*_{11}\ ,\ b_{12} = b^*_{12} \text{ and } b_{22} = b^*_{22}. \qquad (9\_2.20)$$

If we choose p to be positive, then the origin has to be inside S, and we write

$$p > 0. \qquad (9\_2.21)$$

Thus the integrand in the left hand side of (9_2.17) is nonpositive, and thus we have

$$\iint_S H^* dA \leq \iint_S H\ dA. \qquad (9\_2.22)$$

Since the relation between S and S^* is symmetrical we can also have

$$\iint_S H\ dA \leq \iint_S H^* dA. \qquad (9\_2.23)$$

Hence,

$$\iint_S H\ dA = \iint_S H^* dA. \qquad (9\_2.24)$$

Equation (9_2.24) means that

$$\iint_S \left(2K - j\right) p \, dA = 0 \qquad (9\_2.25)$$

which means that $T = 0$, if we are to satisfy $T \leq 0$ and $p > 0$.

Therefore if $T = 0$, equations (9_2.20) are satisfied, and we say that, a complete strictly convex surface is unwarpable which also means it is unbendable. The surfaces S and S^* differ only by an Euclidean motion. This completes the proof of Cohn_Vossen theorem for the rigidity of surfaces with $K>0$.

This proof shows that an ovaloid is unwarpable and therefore rigid. However, it does not obviously prove that the ovaloid is infinitesimally rigid. This will be discussed in section (9_3) where it is proved that the ovaloid is infinitesimally rigid.

A controversial case for the above theorem seems to exist at first sight if one considers the folded sphere in fig.(9_2.1).

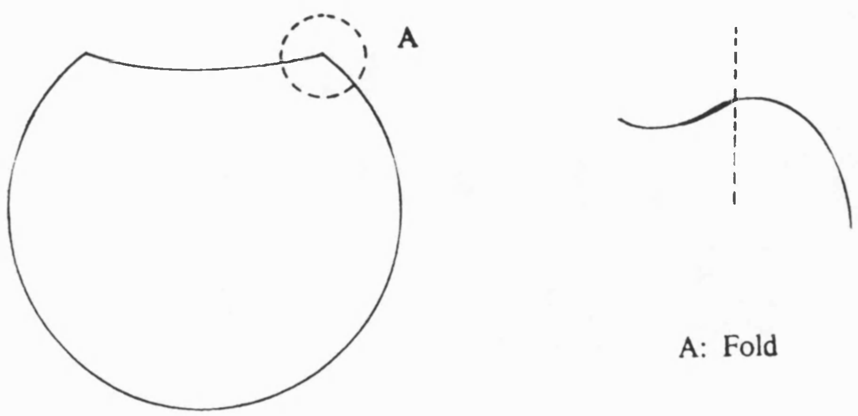


fig.(9_2.1) A sphere with a cap pushed inwards

The folded sphere is isometric to the original sphere and looks as if it contradicts Cohn_Vossen theorem. However, if one examine closely the area where the folds take place, one finds that a real surface cannot practically have a cusp, where the Gaussian curvature becomes infinite, in view of the finite thickness. However, if the fold is not sharp the Gaussian curvature tends to zero at the point where one of the principal curvature start changing the sign and then the surface would not be isometric to the original surface. It is not isometric because the Gaussian curvature was positive and becomes zero after deformation at the fold, which explains that stretching took place and the deformation is not purely inextensional.

9_2.2 The rigidity of closed surfaces with $K \geq 0$

In this class of surfaces a wide range of shells can be obtained by a combination of elliptic, parabolic and planar points on the surface of the shell. Spivak (1979) contains an extension to the rigidity theorem of Cohn_Vossen to cover those surfaces which are convex, as opposed to the strictly convex surfaces in the previous section. As example of these surfaces, we mention the closed cylinder with spherical caps at both ends and the closed spherical belt with discs at the ends. Spivak (1979) starts the analysis by the following:

Equations (9_2.19) and (9_2.21) remain valid in the case of $K \geq 0$, i.e.,

$$T \leq 0 \quad , \quad p > 0.$$

Now, if the curvature tensors $b_{\alpha\beta}$ and $\overset{*}{b}_{\alpha\beta}$ or one of them is zero for all α and β, we have local flat area on the convex surface and therefore equation (9_2.25) holds i.e.,

$$T = 0 \qquad \text{if} \qquad p > 0.$$

The other alternative is that the surface does not contain flat area, then we must use (9_2.18) to find the value of T which in this case cannot be definitely divided by $b_{11}\overset{*}{b}_{11}$ if one of them or both are equal to zero. Note that if the closed surface with $K \geq 0$ does not contain a flat area, then it must have only parabolic points where K is zero. Thus one of the curvature tensors, either in ϑ^1 or ϑ^2 direction is equal to zero. In this case the alternative form for (9_2.18) is

$$T = -\frac{1}{a\, b_{22}\overset{*}{b}_{22}}\left[Ka\left(b_{22} - \overset{*}{b}_{22}\right)^2 + \left(b_{22}\overset{*}{b}_{12} - \overset{*}{b}_{22}b_{12}\right)^2\right]. \qquad (9\_2.26)$$

Thus if K = 0, then T = 0,

1)_if the surface S or S^* or both of them are locally flat

2)_ when

$$\frac{\overset{*}{b}_{11}}{b_{11}} = \frac{\overset{*}{b}_{12}}{b_{12}} = \frac{\overset{*}{b}_{22}}{b_{22}}, \qquad (9\_2.27)$$

which only means that the straight line generators on S and S^* lie in the same directions. For straight lines on the surface the normal curvature vanish and so does the geodesic curvature, the mapping of the generators is geodesic, according to Struik (1961) the geodesic mapping preserves geodesics. The straight line generators on the area where $K = 0$ must start and finish on the edge of an area where $K > 0$ if it is to satisfy $K \geq 0$. Then the curvature in the perpendicular direction to the straight line at the edges on both surfaces S and S^* are equal. Thus, if the curvature in the perpendicular direction to the generators is equal at the edges of the closure, it is equal everywhere on the parabolic surface. This completes the extension of Cohn_Vossen theorem concerning the rigidity of convex surfaces.

Up to this stage, it has been shown that compact convex surfaces are rigid, including those surfaces containing planar points. However, by saying that, it is not meant that these surfaces are infinitesimally rigid. It is also known that, by simply checking the static equations (8_2.4), a locally flat surface cannot support a normal load by membrane action only.

Therefore, it seems that small inextensional deformations are possible if the surface contains planar points even thought finite bending is not. As we are pursuing the aim to design shells that carry loads only by membrane action then we ought to show that these shells are infinitesimally rigid and do not undergo small inextensional deformation. This will make the subject of the next section.

9_3 Infinitesimal rigidity of surfaces

In this section, we are concerned with strictly rigid surfaces where the concept of inextensional deformation plays a decisive role. The discussion follows that given in Efimov (1962).

From equation (8_3.8) we have

$$(\overline{\Omega},_2 \times \mathbf{a}_1) = (\overline{\Omega},_1 \times \mathbf{a}_2) \qquad (9\_3.1)$$

As $\mathbf{a}_\alpha$ are the base vectors, they then lie in the plane of the surface and the two vectors in both sides of equation (9_3.1) lie in the normal direction to the surface. A solution to (9_3.1) is given by (8_4.3) and is

$$\overline{\Omega},_\alpha = \varepsilon^{\rho\vartheta} c_{\alpha\vartheta}\, \mathbf{a}_\rho .$$

From (8_4.7), it was found that the necessary condition to prevent inextensional deformation is to show that the bending strain tensor must be equal to zero.

Scalar multiplying (9_3.1) by $\overline{\Omega}$, we write

$$(\overline{\Omega},_2 \times \mathbf{a}_1).\overline{\Omega} = (\overline{\Omega},_1 \times \mathbf{a}_2).\overline{\Omega} . \qquad (9\_3.2)$$

Equation (9_3.2) can be written as follows

$$\Big[(\mathbf{r} \times \overline{\Omega}).\,\overline{\Omega},_2\Big]_{,1} - \Big[(\mathbf{r} \times \overline{\Omega}).\,\overline{\Omega},_1\Big]_{,2} = 2\Big[\,(\mathbf{r} \times \overline{\Omega},_1).\,\overline{\Omega},_2\Big]. \qquad (9\_3.3)$$

By applying the integral formula of Blaschke which is a specialization of Green theorem, we write

$$\int_C \left[r x \,\overline{\Omega}\right] . \,\overline{\Omega},_2 d\vartheta^2 + \int_C \left[r x \,\overline{\Omega}\right] . \,\overline{\Omega},_1 d\vartheta^1 = 2\iint_S \left[\,\overline{\Omega},_1 x \,\overline{\Omega},_2\right] . r d\vartheta^1 d\vartheta^2 \quad (9\_3.4)$$

which becomes, if we use (8_4.3)

$$= 2\iint_S \varepsilon^{12}\left[(\varepsilon^{\rho\gamma} c_{1\gamma} a_\rho) x (\varepsilon^{\eta\xi} c_{2\xi} a_\eta)\right] . r dA = 2\iint_S \varepsilon^{\alpha\beta}\varepsilon^{\gamma\xi}\, c_{\alpha\gamma} c_{\beta\xi} \,\mathbf{n.r}\, dA.$$

$$\int_C \left[r \; x \,\overline{\Omega}\right] . \,\overline{\Omega},_\alpha d\vartheta^\alpha = 2\iint_S \varepsilon^{\alpha\beta}\varepsilon^{\gamma\xi}\, c_{\alpha\gamma} c_{\beta\xi} \,\mathbf{n.r}\, dA. \qquad (9\_3.5)$$

9_3.1 Strictly convex surfaces (K>0)

In the same manner as before, we start first the analysis of strictly convex surfaces and then extend it, if it is possible, to cover a wider range of surfaces. Efimov (1962) and Spivak (1979) contain a proof for this theorem but in the present analysis it is aimed to proceed differently to allow for engineering concepts.

If the surface is going to be complete compact, strictly convex then the boundary integral on the left hand side of equation (9_3.5) vanish, and we write

$$\iint_S \left[\varepsilon^{\lambda\alpha}\varepsilon^{\mu\beta}\; c_{\lambda\mu} c_{\alpha\beta}\right] \mathbf{n.r}\; dA = 0. \qquad (9\_3.6)$$

For an inextensional deformation of the surface we have from (8_4.7), the following relations

$$\varepsilon^{\lambda\alpha}\varepsilon^{\upsilon\beta}\; b_{\lambda\upsilon} c_{\alpha\beta} = 0 \quad , \quad c_{\alpha\beta} = c_{\beta\alpha} \; , \quad \varepsilon^{\alpha\lambda} c_{\alpha\gamma}|_{\lambda} = 0. \qquad (9\_3.7)$$

If the surface does not contain flat areas, then $c_{\alpha\beta}$ have the following solution

$$c_{\alpha\beta} = \left[U\left[\varepsilon^{\rho\mu}\varepsilon_{\phi\eta} b^{\phi}_{\rho}\; b^{\eta}_{\mu}\; a_{\alpha\beta} - b^{\rho}_{\rho}\; b^{\phi}_{\alpha}\; a_{\phi\beta}\right] + V\; \surd K\left[\varepsilon_{\alpha\gamma} b^{\gamma}_{\beta} + \varepsilon_{\beta\gamma} b^{\gamma}_{\alpha}\right]\right] / \; 2\; R$$

$$c_{\alpha\beta} = U\left[K\; a_{\alpha\beta} - H\; b_{\alpha\beta}\right] \; / \; [R] \; + V\surd K\left[\varepsilon_{\alpha\gamma} b^{\gamma}_{\beta} + \varepsilon_{\beta\gamma} b^{\gamma}_{\alpha}\right] \; / \; [2\; R] \qquad (9\_3.8)$$

where U and V are scalars and R is given by

$$R = \sqrt{\left(b^{\rho}_{\vartheta}\; b^{\vartheta}_{\rho} \,/\, 2 - b^{\nu}_{\nu}\; b^{\eta}_{\eta} \,/ 4 \right)} = \sqrt{\left(H^2 - K\right)}. \qquad (9\_3.9)$$

R represents the radius of Mohr's circle of curvatures.

If we substitute equation (9_3.8) into the first equation of (9_3.7) we get

$$\varepsilon^{\lambda\alpha}\varepsilon^{\upsilon\beta} b_{\lambda\upsilon} c_{\alpha\beta} = U\left[\varepsilon^{\rho\mu}\varepsilon_{\phi\eta}\varepsilon^{\lambda\alpha}\varepsilon^{\upsilon\beta} b_{\lambda\upsilon} b^{\phi}_{\rho}\; b^{\eta}_{\mu}\; a_{\alpha\beta} - \varepsilon^{\lambda\alpha}\varepsilon^{\upsilon\beta} b_{\lambda\upsilon} b^{\rho}_{\rho} b^{\phi}_{\alpha} a_{\phi\beta}\right] / \; [2\; R]$$

$$+ V \sqrt{K} \left[\varepsilon_{\alpha\gamma}\varepsilon^{\lambda\alpha}\varepsilon^{\upsilon\beta}b_{\lambda\upsilon}b^{\gamma}_{\beta} + \varepsilon_{\beta\gamma}\varepsilon^{\lambda\alpha}\varepsilon^{\upsilon\beta}b_{\lambda\upsilon}b^{\gamma}_{\alpha}\right]\Big/ [2\ R]$$

$$\varepsilon^{\lambda\alpha}\varepsilon^{\upsilon\beta}b_{\lambda\upsilon}c_{\alpha\beta}=U\left[\ K\ b^{\phi}_{\phi} - K\ b^{\rho}_{\rho}\right] / R + V\sqrt{K}\left[-\varepsilon^{\upsilon\beta}b_{\lambda\upsilon}b^{\lambda}_{\beta} - \varepsilon^{\lambda\alpha}b_{\lambda\upsilon}b^{\upsilon}_{\alpha}\right]/[2\ R]$$

$$= 0. \qquad (9\_3.10)$$

Now, by assuming that the coordinates follow the line of principal curvatures, consequently $a_{12}= 0$ and $b^{1}_{2} = b^{2}_{1}= 0$, then

$$R= \sqrt{\left(b^{1}_{1}- b^{2}_{2}/2\right)^{2}} = \left(b^{1}_{1}- b^{2}_{2}\right) / 2 \qquad (9\_3.11)$$

and the bending strain components are

$$c_{11}= U\left[2\ b^{1}_{1}b^{2}_{2}\ a_{11} - \left(b^{1}_{1} + b^{2}_{2}\right)\ b^{1}_{1}\ a_{11}\right]/ [2\ R]+ V\sqrt{K}\ [\ 0\ ]$$

$$= U\left[b^{1}_{1}b^{2}_{2}\ a_{11} - b^{1}_{1}b^{1}_{1}\ a_{11}\right]/ [2\ R] = -\ U\ b_{11}$$

$$c_{22}= U\left[2\ b^{1}_{1}b^{2}_{2}\ a_{22} - \left(b^{1}_{1} + b^{2}_{2}\right)\ b^{2}_{2}\ a_{22}\right]/ [2\ R]+ V\sqrt{K}\ [\ 0\ ]$$

$$= U\left[b^{1}_{1}b^{2}_{2}\ a_{22} - b^{2}_{2}b^{2}_{2}\ a_{22}\right]/ [2\ R] = U\ b_{22}$$

$$c_{12}= c_{21}= U\ [\ 0\ ] + V\sqrt{K}\left[\varepsilon_{12}b^{2}_{2} +\varepsilon_{21}b^{1}_{1}\right] / [2\ R] = -\ V\sqrt{K}\ \varepsilon_{12}.$$

$$(9\_3.12)$$

The quantity in brackets in (9_3.6), then is

$$\varepsilon^{\lambda\alpha}\varepsilon^{\mu\beta}\, c_{\lambda\mu}c_{\alpha\beta}=\varepsilon^{\lambda\alpha}\varepsilon^{\mu\beta}\Big\{U^2\left[Ka_{\alpha\beta}\text{-}H\ b_{\alpha\beta}\right]\left[Ka_{\lambda\mu}\text{-}H\ b_{\lambda\mu}\right]\ /\ [R^2]$$

$$+\ UV\ \sqrt{K}\ \left[K\ a_{\alpha\beta}\text{-}\ H\ b_{\alpha\beta}\ \right]\ \left[\varepsilon_{\lambda\gamma}\ b^{\gamma}_{\mu}\ +\ \varepsilon_{\mu\gamma}\ b^{\gamma}_{\lambda}\ \right]\ /\ [2R^2]$$

$$+\ UV\ \sqrt{K}\ \left[K\ a_{\lambda\mu}\text{-}\ H\ b_{\lambda\mu}\ \right]\ \left[\varepsilon_{\alpha\gamma}\ b^{\gamma}_{\beta}\ +\ \varepsilon_{\beta\gamma}\ b^{\gamma}_{\alpha}\ \right]\ /\ [2R^2]$$

$$+\ V^2\ K\ \left[\varepsilon_{\lambda\gamma}\ b^{\gamma}_{\mu}\ +\ \varepsilon_{\mu\gamma}\ b^{\gamma}_{\lambda}\ \right]\ \left[\varepsilon_{\alpha\gamma}\ b^{\gamma}_{\beta}\ +\ \varepsilon_{\beta\gamma}\ b^{\gamma}_{\alpha}\ \right]/\ [4R^2]\Big\}.$$

$$\varepsilon^{\lambda\alpha}\varepsilon^{\mu\beta}\, c_{\lambda\mu}c_{\alpha\beta}=\ U^2\left[\ 2\ K^2\ +\ 2\ H^2K\ -\ 4\ K\ H^2\right]\ /\ R^2$$

$$+\ V^2\ K\ \left[b^{\gamma}_{\gamma}b^{\mu}_{\mu}\ -\ b^{\gamma}_{\mu}\ b^{\mu}_{\gamma}\ \text{-}b^{\gamma}_{\mu}\ b^{\mu}_{\gamma}\ \right]\ /\ [2R^2]$$

$$=\ 2\left[U^2+\ V^2\right]K\ \left[\ K\ -\ H^2\right]\ /\ R^2=\ -\ 2\left[U^2+\ V^2\right]K. \qquad (9\_3.13)$$

Finally, substituting (9_3.13) into (9_3.6), we have for a complete ovaloid the following expression

$$\iint_S K\left[U^2+\ V^2\right]\ \mathbf{n.r}\ dA\ =\ 0. \qquad (9\_3.14)$$

By choice of orientation, the support function **n.r** is positive. Then if the surface is strictly convex the Gaussian curvature is definitely positive and from (9_3.14), we have

$$\left[U^2 + V^2\right] = 0 \qquad (9\_3.15)$$

which means that $U = V = 0$ and consequently from (9_3.12) the strain tensor $c_{\alpha\beta} = 0$ for all α and β. This means that strictly convex surfaces are infinitesimally rigid apart from rigid body motion, by (8_4.3), in surfaces that are not attached to the ground.

It is worth to point out that if the surface is complete with $K < 0$, equation (9_3.14) shows that this surface is also infinitesimally rigid. However, it is physically not possible to have a complete compact surface with negative Gaussian curvature everywhere.

9_3.2 Convex surfaces with ($K \geq 0$)

As we stated before, surfaces that have planar points, i.e. $b_{\alpha\beta} = 0$ for all α and β, are not capable of withstanding the normal component of the surface load by membrane action only. They constitute a mechanism as far as the satisfaction of the static equilibrium is concerned.

For regions where $K = 0$, planar points are excluded and only parabolic points are considered. Hence, if we consider that the lines of generators are parallel to the coordinate ϑ^2 then in our notation in addition to $b_{12} = 0$, we have also $b_{22} = 0$. The curvature tensor in the direction perpendicular to the generators, b_{11}, is different from zero as planar regions are excluded. Equation (9_3.14) is identically satisfied for parabolic regions and so it is for planar regions by virtue of K = 0. However, it is requested to prove in (9_3.14) that U = V = 0, i.e., $c_{\alpha\beta}$ vanish for all α and β in regions where K =0.

If $b_{12} = b_{22} = 0$, then using (9_3.12) we get

$$c_{12} = c_{21} = c_{22} = 0. \tag{9\_3.16}$$

Because of the vanishing of the Gaussian curvature K, the order of covariant differentiation on the parabolic regions of the surface is immaterial. Then, from the third equation of (9_3.7), we write

$$c_{\alpha\beta} = \chi|_{\alpha\beta}. \tag{9\_3.17}$$

Then, using the principle of covariant differentiation for scalar functions, we write

$$c_{22} = \chi|_{22} = \chi_{,22} - \chi_{,\rho}\, \Gamma^{\rho}_{22}. \tag{9\_3.18}$$

For straight line generators and orthogonal coordinates, using (3_3.40) and (3_3.41) we have

$$\begin{aligned} \Gamma^{2}_{22} &= \frac{1}{2} a^{22}\, a_{22,2} \\ \Gamma^{1}_{22} &= \mathbf{a}^{1} . \mathbf{a}_{2,2} = 0. \end{aligned} \tag{9\_3.19}$$

Using (9_3.16) and (9_3.19), equation (9_3.18) then becomes

$$c_{22} = \chi|_{22} = \chi_{,22} - \chi_{,2} \frac{1}{2} a^{22} a_{22,2} = 0$$

$$= \sqrt{(a_{22})}\left[\chi_{,2} / \sqrt{(a_{22})} \right]_{,2} = 0. \qquad (9\_3.20)$$

As we have on the parabolic regions an orthogonal coordinate system and straight line generators, then the coordinate system can be called geodesic coordinate system and the following assumption can be made

$$a_{22} = 1, \qquad (9\_3.21)$$

where equation (9_3.20) then may have the following solution

$$\chi = \phi \; f_1(\vartheta) + f_2(\vartheta), \qquad (9\_3.22)$$

where ϕ is the coordinate along the generators and ϑ is in the perpendicular direction.

Using (3_3.40) and (9_3.21) we write for the geodesic coordinate system the following relations

$$\Gamma^1_{11} = \frac{1}{2} a^{11} a_{11,1} \qquad \Gamma^1_{12} = \frac{1}{2} a^{11} a_{11,2}$$

$$\Gamma^1_{22} = 0 \qquad \Gamma^2_{22} = 0 \qquad (9\_3.23)$$

$$\Gamma^2_{21} = 0 \qquad \Gamma^2_{11} = -\frac{1}{2} a_{11,2}.$$

Equation (9_3.17) (using the covariant differentiation and equation (9_3.22), (9_3.23)) gives

$$c_{11} = \chi|_{11} = \chi_{,11} - \chi_{,1}\Gamma^1_{11} - \chi_{,2}\Gamma^2_{11}$$

$$= \phi \; f_1'' + f_2'' - \left[\phi \, f_1' + f_2'\right] \frac{1}{2 \, a_{11}} a_{11,1} + f_1 \frac{1}{2} a_{11,2}$$

$$c_{12} = \chi|_{12} = \chi_{,12} - \chi_{,1}\Gamma^1_{12} - \chi_{,2}\Gamma^2_{12}$$

$$= f_1' - \left[\phi\, f_1' + f_2'\right] \frac{1}{2a_{11}} a_{11,2}. \qquad (9\_3.24)$$

For parabolic regions on the surface, the Gaussian curvature K vanishes, then using equation (3_3.55) and the geodesic properties, we have

$$K = -\frac{1}{2a_{11}} a_{11,22} + \frac{1}{4(a_{11})^2} (a_{11,2})^2 = 0$$

$$= -\frac{\left[a_{11,2} / 2\surd(a_{11})\right]_{,2}}{\surd(a_{11})} = -\frac{\left[\surd(a_{11})\right]_{,22}}{\surd(a_{11})} = 0. \qquad (9\_3.25)$$

Thus, we can write the following solution

$$\surd(a_{11}) = \phi\, f_3(\vartheta) + f_4(\vartheta), \qquad (9\_3.26)$$

and equations (9_3.24) become

$$c_{11} = \phi\, f_1'''' + f_2'''' - \frac{(\phi\, f_1' + f_2')(\phi f_3' + f_4')}{(\phi f_3 + f_4)} + f_1(\phi f_3 + f_4) f_3$$

$$(9\_3.27)$$

$$c_{12} = f_1' - \left[\phi f_1' + f_2'\right] \frac{f_3}{(\phi f_3 + f_4)} = (f_1' f_4 - f_3 f_2')/(\phi f_3 + f_4).$$

On the elliptic regions of the surface we have already established that the tensor $c_{\alpha\beta}$ vanishes for all α and β. Therefore it also vanishes at the end of the generators of the parabolic regions, and with (9_3.16) the second equation of (9_3.27) we get

$$c_{12} = (f_1' f_4 - f_3 f_2')/(\phi f_3 + f_4) = 0$$

$$(f_1' f_4 - f_3 f_2') = 0. \tag{9\_3.28}$$

Substituting this result into the first equation of (9_3.27) shows that for certain value of ϑ, c_{11} is a linear function of ϕ. Therefore by knowing that c_{11} is also zero at two values of ϕ (the generator ends), then c_{11} must be zero for all ϕ. Adding this result to (9_3.16) we conclude that $c_{\alpha\beta}$ vanishes on the parabolic regions for all α and β. Finally we say that a complete convex surface with $K \geq 0$ free from planar points is infinitesimally rigid.

9_3.3 Floating surface with a hole

The compactness of surfaces in the foregoing sections was a necessary condition for studying rigidity. Now, consider a floating surface, i.e. not attached to the foundation, with a loaded hole as shown in fig.(9_3.1). The loads lie in the local plane of the surface.

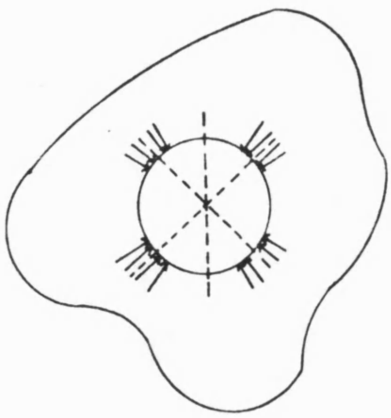


fig.(9_3.1) Floating surface with a hole

The load being self_balanced cannot do any work when the shell undergoes a rigid body motion. The load on each area separately is not in equilibrium. If the structure is not a mechanism the boundary loads will be balanced by an internal state of membrane stress represented by $n^{\rho\gamma}$. However, around the hole we can also have a deformation defined by $c_{\alpha\beta}$ such that

$$c_{\alpha\beta}= \varepsilon_{\alpha\rho}\varepsilon_{\beta\gamma}n^{\rho\gamma}.$$

Thus a floating surface with a hole must be a mechanism.

9_4 Conclusion

In this chapter the general rules for designing membrane shells using the concept of inextensional deformation have been investigated.

It was found that convex surfaces with K (the Gaussian curvature) ≥0 are infinitesimally rigid provided planar points are excluded. Surfaces with planar points admit inextensional deformation and constitute a mechanism when subjected to normal external load.

A convex surface such as a hemisphere fixed to a foundation must be rigid since the restraint produced by the foundation must be at least as effective as the restraint provided by the missing part of the sphere.

It was also found that floating surfaces with holes are mechanisms.

The following table summarises the different results of the present chapter.

	$K>0$ Strictly convex	$K\geq 0$ Convex	$K\geq 0$ Convex with planar points	any K Floating with a hole
Infinitesimally rigid	yes	yes	no	no
Finitely rigid	yes	yes	yes	?
Mechanism	no	no	yes	yes

Table (3): Rigidity of surfaces of different K

CHAPTER TEN

THE INEXTENSIONAL DEFORMATION IN SHELLS OF REVOLUTION

10_1 Introduction

A surface of revolution is characterized by rotating a plane curve, say c, about an axis A which lies in its plane, fig.(10_1.1). The plane curve c will be called the meridian curve, and a point P on the curve c when shifted in a direction perpendicular to the axis A and tangential to the meridian curve c will trace out another curve perpendicular to c called the parallel circle (latitude circle). Then, the axis A will be called the axis of revolution and the surface produced in this manner is a surface of revolution.

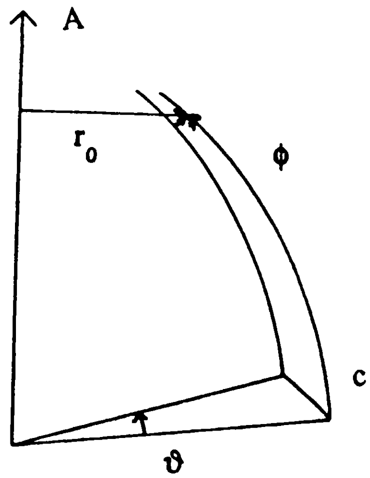


fig. (10_1.1) surface of revolution

If we consider that the position of the parallels in the surface are defined by the coordinate $r_0 = r_0(\phi)$, where r_0 is the radius of the parallel circle at the position ϕ then, from fig.(10_1.1) the position vector of the point P on the surface will be

$$\mathbf{r}(\phi, \vartheta) = r_0(\phi)\cos\vartheta\ \mathbf{i} + r_0(\phi)\sin\vartheta\ \mathbf{j} + z(\phi)\ \mathbf{k} \tag{10\_1.1}$$

where

$$\left.\begin{aligned} r_0 &= f(\phi) \\ z &= z(\phi) \end{aligned}\right\}. \tag{10\_1.2}$$

Thus, from (3_3.3) the base vectors are

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial \vartheta^\alpha} = \mathbf{r}_{,\alpha},$$

where the index α takes the values 1 and 2, and in this case it represents the coordinates ϑ and ϕ respectively. Hence,

$$\left.\begin{aligned} \mathbf{a}_1 &= -\,r_0 \sin\vartheta\ \mathbf{i} + r_0 \cos\vartheta\ \mathbf{j} \\ \mathbf{a}_2 &= r_0' \cos\vartheta\ \mathbf{i} + r_0' \sin\vartheta\ \mathbf{j} + z'\ \mathbf{k} \end{aligned}\right\} \tag{10\_1.3}$$

where the dash stands for derivative with respect to ϕ,

$$\left.\begin{aligned} r_0' &= \frac{\partial r_0}{\partial \phi} \\ z' &= \frac{\partial z}{\partial \phi} \end{aligned}\right\}. \tag{10\_1.4}$$

From (3_3.8) and (10_1.3), using the scalar product we can write the surface metrics

$$\left.\begin{aligned} \mathbf{a}_1 \cdot \mathbf{a}_1 &= a_{11} = r_0^2 \\ \mathbf{a}_1 \cdot \mathbf{a}_2 &= a_{12} = 0 \\ \mathbf{a}_2 \cdot \mathbf{a}_2 &= a_{22} = r_0'^2 + z'^2 \end{aligned}\right\}. \tag{10\_1.5}$$

The second equation shows that the meridians and parallels form an orthogonal family of parametric lines.

Also from (3_3.11), the determinant of the first fundamental form will be given the symbol A to avoid confusion with the radius a and will be

$$A = |a_{\alpha\beta}| = a_{11}a_{22} - a_{12}^2,$$

with $a_{12} = 0$

$$A = r_0^2\,(r_0'^2 + z'^2). \qquad (10\_1.6)$$

Equation (3_3.10) reduces to the following since the coordinates are orthogonal

$$\left.\begin{aligned} a^{11} &= \frac{1}{a_{11}} = \frac{1}{r_0^2} \\ a^{12} &= a^{21} = 0 \\ a^{22} &= \frac{1}{a_{22}} = \frac{1}{r_0'^2 + z'^2} \end{aligned}\right\}. \qquad (10\_1.7)$$

From (3_3.12) and $a^{12} = 0$, we write the contravariant base vectors

$$\begin{aligned} \mathbf{a}^1 &= a^{11}\,\mathbf{a}_1 = -\frac{1}{r_0}\Big[\sin\vartheta\,\mathbf{i} - \cos\vartheta\,\mathbf{j}\Big] \\ \mathbf{a}^2 &= a^{22}\,\mathbf{a}_2 = \frac{1}{r_0'^2 + z'^2}\Big[r_0'\cos\vartheta\,\mathbf{i} + r_0'\sin\vartheta\,\mathbf{j} + z'\,\mathbf{k}\Big]. \end{aligned} \qquad (10\_1.8)$$

Equation (3_3.6) gives the normal to the surface

$$(\mathbf{a}_1 \times \mathbf{a}_2) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r_0\sin\vartheta & r_0\cos\vartheta & 0 \\ r_0'\cos\vartheta & r_0'\sin\vartheta & z' \end{bmatrix}$$

$$(\mathbf{a}_1 \times \mathbf{a}_2) = \; r_0 z' \cos\vartheta\, \mathbf{i} + r_0 z' \sin\vartheta\, \mathbf{j} - r_0 r_0' \mathbf{k}.$$

The magnitude of the above base vector, when $\mathbf{a}_1$ and $\mathbf{a}_2$ are orthogonal i.e $\gamma = 90^\circ$ and using (3_2.51), is

$$|\mathbf{a}_1 \times \mathbf{a}_2| = |\mathbf{a}_1|.|\mathbf{a}_2| \sin\gamma$$

$$= \sqrt{(a_{11} a_{22})} = \sqrt{A} = \sqrt{\left[r_0^2 (r_0'^2 + z'^2) \right]}.$$

Therefore, the normal to the surface in equation (3_3.6), reads

$$\mathbf{n} = \frac{1}{\sqrt{(r_0'^2 + z'^2)}} \left[z' \cos\vartheta\, \mathbf{i} + z' \sin\vartheta\, \mathbf{j} - r_0' \mathbf{k} \right]. \qquad (10\_1.9)$$

The derivatives of the base vectors are

$$\left.\begin{aligned} \mathbf{a}_{1,1} &= - r_0 \cos\vartheta\, \mathbf{i} - r_0 \sin\vartheta \mathbf{j} \\ \mathbf{a}_{2,2} &= r_0'' \cos\vartheta\, \mathbf{i} + r_0'' \sin\vartheta \mathbf{j} + z'' \mathbf{k} \\ \mathbf{a}_{1,2} &= \mathbf{a}_{2,1} = - r_0' \sin\vartheta\, \mathbf{i} + r_0' \cos\vartheta\, \mathbf{j} \end{aligned}\right\}. \qquad (10\_1.10)$$

These derivatives allow us to evaluate the curvature tensors, then from equation (3_3.26) we have

$$\left.\begin{aligned} b_{11} &= \mathbf{a}_{1,1}.\mathbf{n} = - \frac{r_0 z'}{\sqrt{(r_0'^2 + z'^2)}} \\ b_{22} &= \mathbf{a}_{2,2}.\mathbf{n} = \frac{1}{\sqrt{(r_0'^2 + z'^2)}} \left[r_0'' z' - z'' r_0' \right] \end{aligned}\right. \qquad (10\_1.11)$$

$$b_{12} = b_{21} = 0.$$

Using (3_3.27), the mixed curvature tensors are

$$b^1_1 = - \frac{z'}{r_0\sqrt{(r_0'^2 + z'^2)}}$$

$$b^2_2 = \frac{1}{(r_0'^2 + z'^2)^{3/2}} \left[r_0'' z' - z'' r_0' \right] \tag{10_1.12}$$

$$b^2_1 = b^1_2 = 0.$$

Thus, from (3_3.37) and (3_3.38), the mean and Gaussian curvatures are

$$2\,H = \frac{-\,z'(r_0'^2 + z'^2) + r_0(r_0'' z' - z'' r_0')}{r_0\,(r_0'^2 + z'^2)^{3/2}}$$

$$K = - \frac{z'(r_0'' z' - z'' r_0')}{r_0\,(r_0'^2 + z'^2)^2}. \tag{10_1.13}$$

The Christoffels for the surface of revolution are from (3_3.41)

$$\Gamma^1_{11} = \mathbf{a}^1.\mathbf{a}_{1,1} = 0 \qquad \Gamma^2_{22} = \mathbf{a}^2.\mathbf{a}_{2,2} = \frac{r_0' r_0'' + z' z''}{(r_0'^2 + z'^2)}$$

$$\Gamma^1_{21} = \mathbf{a}^1.\mathbf{a}_{2,1} = \frac{r_0'}{r_0} \qquad \Gamma^2_{12} = \mathbf{a}^2.\mathbf{a}_{1,2} = 0 \tag{10_1.14}$$

$$\Gamma^1_{22} = \mathbf{a}^1.\mathbf{a}_{2,2} = 0 \qquad \Gamma^2_{11} = \mathbf{a}^2.\mathbf{a}_{1,1} = - \frac{r_0 r_0'}{(r_0'^2 + z'^2)}.$$

Equation (3_3.21) gives the ε_system for the surface of revolution

$$\varepsilon_{12} = -\,\varepsilon_{21} = \sqrt{A} = r_0\sqrt{(r_0'^2 + z'^2)}$$

$$\varepsilon^{12} = -\,\varepsilon^{21} = 1/\sqrt{A} = (\,1/\,(r_0\sqrt{(r_0'^2 + z'^2)})). \tag{10_1.15}$$

The covariant derivative of the Gaussian curvature K is;

$$K|_{\alpha} = \frac{b_{11}|_{\alpha}\, b_{22} + b_{11} b_{22}|_{\alpha} - 2 b_{12} b_{12}|_{\alpha}}{A}. \tag{10_1.16}$$

We write also for further use

$$\Omega|_{\alpha\beta} = \Omega_{,\alpha\beta} - \Gamma^{\gamma}_{\alpha\beta}\, \Omega_{,\gamma} \tag{10_1.17}$$

Then

$$\Omega|_{11} = \Omega_{,11} - \Gamma^{1}_{11}\, \Omega_{,1} - \Gamma^{2}_{11}\, \Omega_{,2}$$

$$\Omega|_{21} = \Omega|_{12} = \Omega_{,12} - \Gamma^{1}_{12}\, \Omega_{,1} - \Gamma^{2}_{12}\, \Omega_{,2}$$

$$\Omega|_{22} = \Omega_{,22} - \Gamma^{1}_{22}\, \Omega_{,1} - \Gamma^{2}_{22}\, \Omega_{,2}.$$

Also from (3_3.50), we can write

$$b_{\alpha\beta}|_{\gamma} = b_{\alpha\beta,\gamma} - \Gamma^{\rho}_{\alpha\gamma}\, b_{\rho\beta} - \Gamma^{\rho}_{\beta\gamma}\, b_{\alpha\rho}. \tag{10_1.18}$$

Now, using equation (8_3.16), we derive the tangential components of the angular velocity as follows

$$\Omega^{\eta} = - \frac{\Omega|_{\beta}\, \varepsilon^{\alpha\eta} \varepsilon^{\gamma\beta} b_{\alpha\gamma}}{K}. \tag{10_1.19}$$

Then, with $b_{12} = b_{21} = 0$ in mind (the coordinates follow the lines of curvature) and $\Omega|_{\alpha} = \Omega_{,\alpha}$, we write

$$\Omega^{1} = - \frac{\Omega|_{1}\, \varepsilon^{12} \varepsilon^{12} b_{22}}{K}, \quad \Omega^{2} = - \frac{\Omega|_{2}\, \varepsilon^{12} \varepsilon^{12} b_{11}}{K}. \tag{10_1.20}$$

Considering the boundary circles running along the circles of latitude, then $\vartheta^2 =$ constant and $d\vartheta^2 = 0$. Thus the rate of change of the normal curvature from (8_4.19) will be

$$k_n^{\cdot} = \frac{(\varepsilon_{12})^2\, \beta^{22} (d\vartheta^1)^2}{a_{11}\,(d\vartheta^1)^2} = \frac{(\varepsilon_{12})^2\, \beta^{22}}{a_{11}}\,, \tag{10\_1.21}$$

and the rate of change of the twist from (8_4.21) will be

$$\tau^{\cdot} = - \frac{(\varepsilon_{12})^3\, \beta^{12} a^{22}}{a_{11}}. \tag{10\_1.22}$$

From equations (5_6.2), the tensor $\beta^{\alpha\beta}$ gives

$$\begin{aligned}
\beta^{12} &= \varepsilon^{12}\Big[\, \Omega^2|_2 - \Omega\, b_2^2 \,\Big] \\
\beta^{21} &= \varepsilon^{21}\Big[\, \Omega^1|_1 - \Omega\, b_1^1 \,\Big] \\
\beta^{22} &= \varepsilon^{21}\Big[\, \Omega^2|_1 - \Omega\, b_1^2 \,\Big] \\
\beta^{11} &= \varepsilon^{12}\Big[\, \Omega^1|_2 - \Omega\, b_2^1 \,\Big] .
\end{aligned} \tag{10_1.23}$$

From equation (8_3.17), we have

$$\Omega^{\eta}|_{\lambda} = - \Big[\ \varepsilon^{\alpha\eta}\, \varepsilon^{\gamma\beta} b_{\alpha\gamma}\ \frac{\Omega|_{\beta}}{K} \Big]\Big|_{\lambda}, \tag{10\_1.24}$$

and then

$$\begin{aligned}
\Omega^1|_1 &= - \Big[\varepsilon^{21}\, \varepsilon^{21} b_{22}\ \frac{\Omega|_1}{K} + \varepsilon^{21}\, \varepsilon^{12} b_{21}\ \frac{\Omega|_2}{K} \Big]\Big|_1 \\
\Omega^1|_2 &= - \Big[\varepsilon^{21}\, \varepsilon^{21} b_{22}\ \frac{\Omega|_1}{K} + \varepsilon^{21}\, \varepsilon^{12} b_{21}\ \frac{\Omega|_2}{K} \Big]\Big|_2 \\
\Omega^2|_1 &= - \Big[\varepsilon^{12}\, \varepsilon^{12} b_{11}\ \frac{\Omega|_2}{K} + \varepsilon^{12}\, \varepsilon^{21} b_{12}\ \frac{\Omega|_1}{K} \Big]\Big|_1 \\
\Omega^2|_2 &= - \Big[\varepsilon^{12}\, \varepsilon^{12} b_{11}\ \frac{\Omega|_2}{K} + \varepsilon^{12}\, \varepsilon^{21} b_{12}\ \frac{\Omega|_1}{K} \Big]\Big|_2
\end{aligned} \tag{10_1.25}$$

10_2 The spherical shell

The parallel circles on a spherical shell of revolution are given by

$$\left.\begin{aligned} r_0 &= a / \cosh\phi \\ z &= r_0 \sinh\phi \end{aligned}\right\} \qquad (10\_2.1)$$

where a is the radius of curvature of the spherical surface.

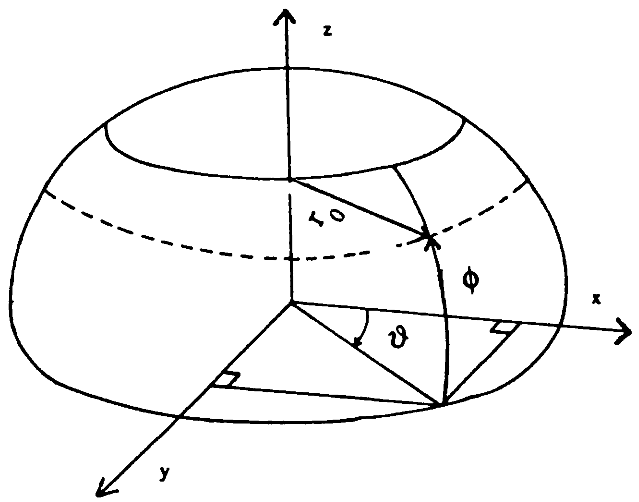


fig.(10_2.1) The spherical shell

Then, the position vector **r**, fig.(10_2.1), of a point on the spherical surface is given by

$$\mathbf{r} = \frac{a}{\cosh\phi}\left[\cos\vartheta\ \mathbf{i} + \sin\vartheta\ \mathbf{j} + \sinh\phi\ \mathbf{k}\right]. \qquad (10\_2.2)$$

The derivatives of r_0 and z with respect to ϕ are

$$r_0' = -\frac{a\sinh\phi}{\cosh^2\phi} \qquad r_0'' = -a\frac{(1-\sinh^2\phi)}{\cosh^3\phi}$$

$$z' = \frac{a}{\cosh^2\phi} \qquad z'' = -\frac{2a\sinh\phi}{\cosh^3\phi}$$

$$\varepsilon_{12} = -\varepsilon_{21} = \sqrt{A} = r_0.\sqrt{(r_0'^2 + z'^2)} = \frac{a^2}{\cosh^2\phi}.$$

The base vectors are

$$\mathbf{a}_1 = \frac{a}{\cosh\phi}\left[-\sin\vartheta\ \mathbf{i} + \cos\vartheta\ \mathbf{j}\right]$$

$$\mathbf{a}_2 = \frac{a}{\cosh^2\phi}\left[-\cos\vartheta\sinh\phi\ \mathbf{i} - \sin\vartheta\ \sinh\phi\ \mathbf{j} + \mathbf{k}\right].$$

The metrics are

$$a_{11} = \frac{a^2}{\cosh^2\phi} \qquad a_{22} = \frac{a^2}{\cosh^2\phi} \qquad a_{12} = a_{21} = 0$$

$$a^{11} = \frac{\cosh^2\phi}{a^2} \qquad a^{22} = \frac{\cosh^2\phi}{a^2} \qquad a^{12} = a^{21} = 0.$$

Then, the contravariant base vectors are

$$\mathbf{a}^1 = \frac{\cosh\phi}{a}\left[-\sin\vartheta\ \mathbf{i} + \cos\vartheta\ \mathbf{j}\right]$$

$$\mathbf{a}^2 = \frac{1}{a}\left[-\cos\vartheta\sinh\phi\ \mathbf{i} - \sin\vartheta\ \sinh\phi\ \mathbf{j} + \mathbf{k}\right].$$

The normal to the surface is

$$\mathbf{n} = \frac{1}{\cosh\phi}\left[\cos\vartheta\ \mathbf{i} + \sin\vartheta\ \mathbf{j} + \sinh\phi\ \mathbf{k}\right].$$

The derivatives of the base vectors are

$$a_{1,1} = -\frac{a}{\cosh\phi}\left[\cos\vartheta\ \mathbf{i} + \sin\vartheta\ \mathbf{j}\right]$$

$$a_{2,2} = -\frac{a}{\cosh^3\phi}\left[\left[1-\sinh^2\phi\right]\left[\cos\vartheta\ \mathbf{i} + \sin\vartheta\mathbf{j}\right] + 2\sinh\phi\ \mathbf{k}\right]$$

$$a_{1,2} = a_{2,1} = \frac{a\sinh\phi}{\cosh^2\phi}\left[\sin\vartheta\ \mathbf{i} - \cos\vartheta\ \mathbf{j}\right].$$

The curvature tensors are

$$b_{22} = b_{11} = -\frac{a}{\cosh^2\phi} \qquad b^1_1 = b^2_2 = -\frac{1}{a}$$

$$b_{12} = b_{21} = 0 \qquad b^1_2 = b^2_1 = 0.$$

The Christoffels for the spherical surface are

$$\Gamma^1_{11} = 0 \qquad \Gamma^2_{22} = -\tanh\phi$$

$$\Gamma^1_{21} = -\tanh\phi \qquad \Gamma^2_{12} = 0$$

$$\Gamma^1_{22} = 0 \qquad \Gamma^2_{11} = \tanh\phi\ .$$

The Gaussian and mean curvatures are from (10_1.13), and they are constants in this particular case

$$K = \frac{1}{a^2} \qquad H = -\frac{1}{a}.$$

Thus

$$K|_\alpha = 0. \qquad (10\_2.3)$$

From equation (8_3.19), we write

$$\varepsilon^{\xi\lambda}\varepsilon^{\rho\gamma}b_{\xi\rho}\left[\frac{\Omega\,|_{\gamma}}{K}\right]\Bigg|_{\lambda}+\Omega\, b^{\lambda}_{\lambda}=0. \qquad (10\_2.4)$$

This represents a single tensor equation, then summation over repeated indices must be taken. As ε^{11} and ε^{22} are both equal to zero, and in this special case $b_{12} = b_{21} = 0$, then we write

$$\varepsilon^{12}\varepsilon^{12}\left[b_{11}\left[\frac{\Omega\,|_{2}}{K}\right]\Bigg|_{2}+b_{22}\left[\frac{\Omega\,|_{1}}{K}\right]\Bigg|_{1}\right]+\Omega\left[b^{1}_{1}+b^{2}_{2}\right]=0 \qquad (10\_2.5)$$

It is valid for all shells of revolution provided that the Gaussian curvature is different from zero, and the coordinates follow the lines of curvature.

Performing the differentiation and using (10_2.3) for the spherical shell, we write

$$\varepsilon^{12}\varepsilon^{12}\left[b_{11}\left[\frac{\Omega\,|_{22}}{K}\right]+b_{22}\left[\frac{\Omega\,|_{11}}{K}\right]\right]+\Omega\left[b^{1}_{1}+b^{2}_{2}\right]=0.$$

Replacing the corresponding geometrical quantities in this particular spherical case and using (10_1.17), we finally write

$$\cosh^{2}\phi\left[\Omega,_{22}+\Omega,_{11}\right]+2\,\Omega=0. \qquad (10\_2.6)$$

Examination of equation (10_2.6) shows that, it is a second order partial differential equation in the normal component of the angular velocity, and according to Berg & McGregor (1969) and Andrews (1986) it is of elliptic type. As we are going to be concerned with shells of revolution in which the edge circle(s) is continuous and closed, the solution of the PDE will be periodic with a period of 2π.

The periodicity of the solution can be made to reduce our problem to a second order ordinary differential equation.

If we write the solution in the following form

$$\Omega = \sum_{n=0}^{\infty} \left[f_n(\phi) \cos n\vartheta + g_n(\phi) \sin n\vartheta \right] \tag{10_2.7}$$

where $f_n(\phi)$ and $g_n(\phi)$ are function of ϕ only, then

$$\Omega,_{11} = - n^2\Omega.$$

Equation (10_2.6) becomes

$$\cosh^2\phi \; f_n''(\phi) \;\; -f_n(\phi) \; (n^2\cosh^2\phi \; - 2)=0$$

$$\cosh^2\phi \; g_n''(\phi) \;\; - g_n(\phi) \; (n^2\cosh^2\phi - 2)=0.$$

These equations have the general solutions

$$f_n(\phi)=A_n\left[n\cosh \, n\phi \; - \sinh \, n\phi \; \tanh \, \phi\right]+B_n\left[n\sinh \, n\phi \; - \cosh \, n\phi \; \tanh \, \phi\right]$$

$$\tag{10_2.8}$$

$$g_n(\phi)=C_n\left[n\cosh \, n\phi \; - \sinh \, n\phi \; \tanh \, \phi\right]+D_n\left[n\sinh \, n\phi \; - \cosh \, n\phi \; \tanh \, \phi\right]$$

where A_n,B_n, C_n,D_n are constants which will be subject to boundary conditions, C_0 and D_0 do not produce a solution since $\sin n\vartheta$ is zero.

The discussion of the special case of the solution when n takes the two values 0 and 1 will be given when we derive the corresponding velocities.

Substituting (10_2.8) into the solution for the normal component of the angular velocity in (10_2.7), we write

$$\Omega = \sum_{n=0}^{\infty}\left[\left(A_n\left[n\cosh n\phi - \sinh n\phi\ \tanh\phi\right] + B_n\left[n\sinh n\phi - \cosh n\phi\ \tanh\phi\right]\right)\cos n\vartheta\right.$$

$$\left.+\left(C_n\left[n\ \cosh n\phi - \sinh n\phi\ \tanh\phi\right] + D_n\left[n\ \sinh n\phi - \cosh n\phi\ \tanh\phi\right]\right)\sin n\vartheta\right].$$

Finally, from (10_1.20) the tangential components are

$$\Omega^1 = n\ \frac{\cosh^2\phi}{a}\sum_{=0}^{\infty}\left[\right.$$

$$-\left(A_n\left[\ n\cosh n\phi\ -\ \sinh n\phi\ \tanh\phi\right] + B_n\left[n\ \sinh n\phi\ -\ \cosh n\phi\ \tanh\phi\right]\right)\sin n\vartheta$$

$$\left.+\left(C_n\left[n\ \cosh n\phi - \sinh n\phi\ \tanh\phi\right] + D_n\left[n\sinh n\phi - \cosh n\phi\ \tanh\phi\right]\right)\cos n\vartheta\right]$$

$$\Omega^2 = \frac{\cosh^2\phi}{a}\ \sum_{=0}^{\infty}\left[\ \left(A_n\left[\sinh n\phi\ \frac{(n^2\cosh^2\phi\ -1)}{\cosh^2\phi} - n\ \cosh n\phi\ \tanh\phi\ \right]\ +\right.\right.$$

$$\left. + \ B_n\left[\cosh n\phi\ \frac{(n^2\cosh^2\phi\ -1)}{\cosh^2\phi} - n\ \sinh n\phi\ \tanh\phi\ \right]\ \right)\cos n\vartheta$$

$$+ \left\{ C_n\left[\sinh n\phi \, \frac{(n^2\cosh^2\phi - 1)}{\cosh^2\phi} - n \cosh n\phi \tanh\phi \right] + \right.$$

$$\left. D_n\left[\cosh n\phi \, \frac{(n^2\cosh^2\phi - 1)}{\cosh^2\phi} - n \sinh n\phi \tanh\phi \right] \right\} \sin n\vartheta \Bigg]. \qquad (10\_2.9)$$

After having found the components of the angular velocity vector of the spherical shell, in the inextensional modes of deformation, the corresponding velocities will be used for the generation of the inextensional modes of deformation, thus from (8_3.3) we have

$$\mathbf{v}_{,\alpha} = \boldsymbol{\Omega} \times \mathbf{a}_{\alpha}, \quad \mathbf{v}_{,1} = \left(\Omega^1\mathbf{a}_1 + \Omega^2\mathbf{a}_2 + \Omega\, \mathbf{n} \right) \times \mathbf{a}_1 .$$

Upon using (3_3.23), we write

$$\mathbf{v}_{,1} = \sqrt{A}\left(- \Omega^2\mathbf{n} + \Omega\, \mathbf{a}^2 \right).$$

Substituting the values of the angular velocities and using the base vectors, we end up with the following expression

$$\mathbf{v}_{,1} = - \frac{a}{\cosh\phi} \sum_{n=0}^{\infty} \left[\left[A_n\left(\left(n^2-1\right) \sinh n\phi \right) + B_n\left(\left(n^2-1\right) \cosh n\phi \right) \right] \cos n\vartheta \right.$$

$$\left. + \left[C_n\left(\left(n^2-1\right) \sinh n\phi \right) + D_n\left(\left(n^2-1\right) \cosh n\phi \right) \right] \sin n\vartheta \right] \times \left(\cos\vartheta\, \mathbf{i} + \sin\vartheta\, \mathbf{j} \right)$$

$$+an\left[\left[A_n\left(-n\sinh n\phi\ \tanh\phi+\cosh\phi\right)+B_n\left(-n\cosh n\phi\ \tanh\phi+\sinh n\phi\right)\right]\cos n\vartheta\right.$$

$$\left.\left[C_n\left(-n\sinh n\phi\ \tanh\phi+\cosh\phi\right)+D_n\left(-n\cosh n\phi\ \tanh\phi+\sinh n\phi\right)\right]\sin n\vartheta\right]\ \mathbf{k}.$$

Then, integrating in the parallel circle's direction, we get

$$\mathbf{v} = -\frac{a}{\cosh\phi}\sum_{n=0}^{\infty}\left[n\left\{\left[A_n\sinh n\phi + B_n\cosh n\phi\right]\sin n\vartheta\right.\right.$$

$$\left.-\left[C_n\sinh n\phi + D_n\cosh n\phi\right]\cos n\vartheta\right\} \times \left(\cos\vartheta\ \mathbf{i} + \sin\vartheta\ \mathbf{j}\right)$$

$$-\left\{\left[A_n\sinh n\phi+B_n\cosh n\phi\right]\cos n\vartheta+\left[C_n\sinh n\phi+D_n\cosh n\phi\right]\sin n\vartheta\right\}\times\left(\sin\vartheta\mathbf{i}-\cos\vartheta\mathbf{j}\right)$$

$$\left[\left\{\left[A_n\sinh n\phi + B_n\cosh n\phi\right]n\sinh\phi - \left[A_n\cosh n\phi + B_n\sinh n\phi\right]\cosh\phi\right\}\sin n\vartheta\right.$$

$$\left.\left.-\left\{\left[C_n\sinh n\phi + D_n\cosh n\phi\right]n\sinh\phi - \left[C_n\cosh n\phi + D_n\sinh n\phi\right]\cosh\phi\right\}\cos n\vartheta\right]\mathbf{k}\right]+\mathbf{c}$$

where the constant vector **c** represents only a translation (rigid body translation).

From the above velocity vector, two kind of deformation can be distinguished:

First, consider the case of n = 0, we end up only with a rotation about the axis Z represented by B_0 in the following expression

$$\frac{a}{\cosh\phi} B_0 \left[\sin\vartheta \, \mathbf{i} - \cos\vartheta \, \mathbf{j}\right].$$

Also the displacement due to n = 1 is

$$-\frac{a}{\cosh\phi}\left[\left[A_1 \sinh\phi + B_1 \cosh\phi\right]\mathbf{j} - \left[C_1 \sinh\phi + D_1 \cosh\phi\right]\mathbf{i} - \left[A_1 \sin\vartheta + C_1 \cos\vartheta\right]\mathbf{k}\right]$$

Examination of this equation reveals that both B_1 and D_1 produce rigid body velocities and A_1, C_1 produce a rigid body angular velocities about the axes X and Y. Therefore the values 0 and 1 of n in the solution will not be considered to belong to the inextensional modes of deformation, and represent only rigid body motion.

Secondly, values of n which are greater or equal to 2 produce a real inextensional deformation which, if not prevented, render the structure mechanism. If the velocity is to be finite at the north pole of the spherical shell when ϕ tends to ∞, then $A_n = - B_n$ and $C_n = - D_n$. Similarly if the velocity is to be finite when ϕ tends to $- \infty$ at the south pole, then $A_n = B_n$ and $C_n = D_n$.

Consequently we say that a complete spherical shell is incapable of undergoing inextensional deformations i.e the complete sphere cannot bend without stretching and shearing of its middle surface.

The components of the velocity vector are

$$v^1 = -\sum_{n=0}^{\infty} \left\{ \left[A_n \sinh n\phi + B_n \cosh n\phi \right] \cos n\vartheta + \left[C_n \sinh n\phi + D_n \cosh n\phi \right] \sin n\vartheta \right\}$$

$$v^2 = \sum_{n=0}^{\infty} \left\{ \left[A_n \cosh n\phi + B_n \sinh n\phi \right] \sin n\vartheta - \left[C_n \cosh n\phi + D_n \sinh n\phi \right] \cos n\vartheta \right\}$$

$$v^3 = \sum_{n=0}^{\infty} \frac{a \sinh\phi}{\cosh\phi} \left\{ \left[A_n \cosh n\phi + B_n \sinh n\phi \right] \sin n\vartheta - \left[C_n \cosh n\phi + D_n \sinh n\phi \right] \cos n\vartheta \right\}$$

$$- an \left\{ \left[A_n \sinh n\phi + B_n \cosh n\phi \right] \sin n\vartheta - \left[C_n \sinh n\phi + D_n \cosh n\phi \right] \cos n\vartheta \right\}.$$

These components will be used to draw the inextensional modes of deformation corresponding to values of n that are greater than 1, fig.(10_2.2)

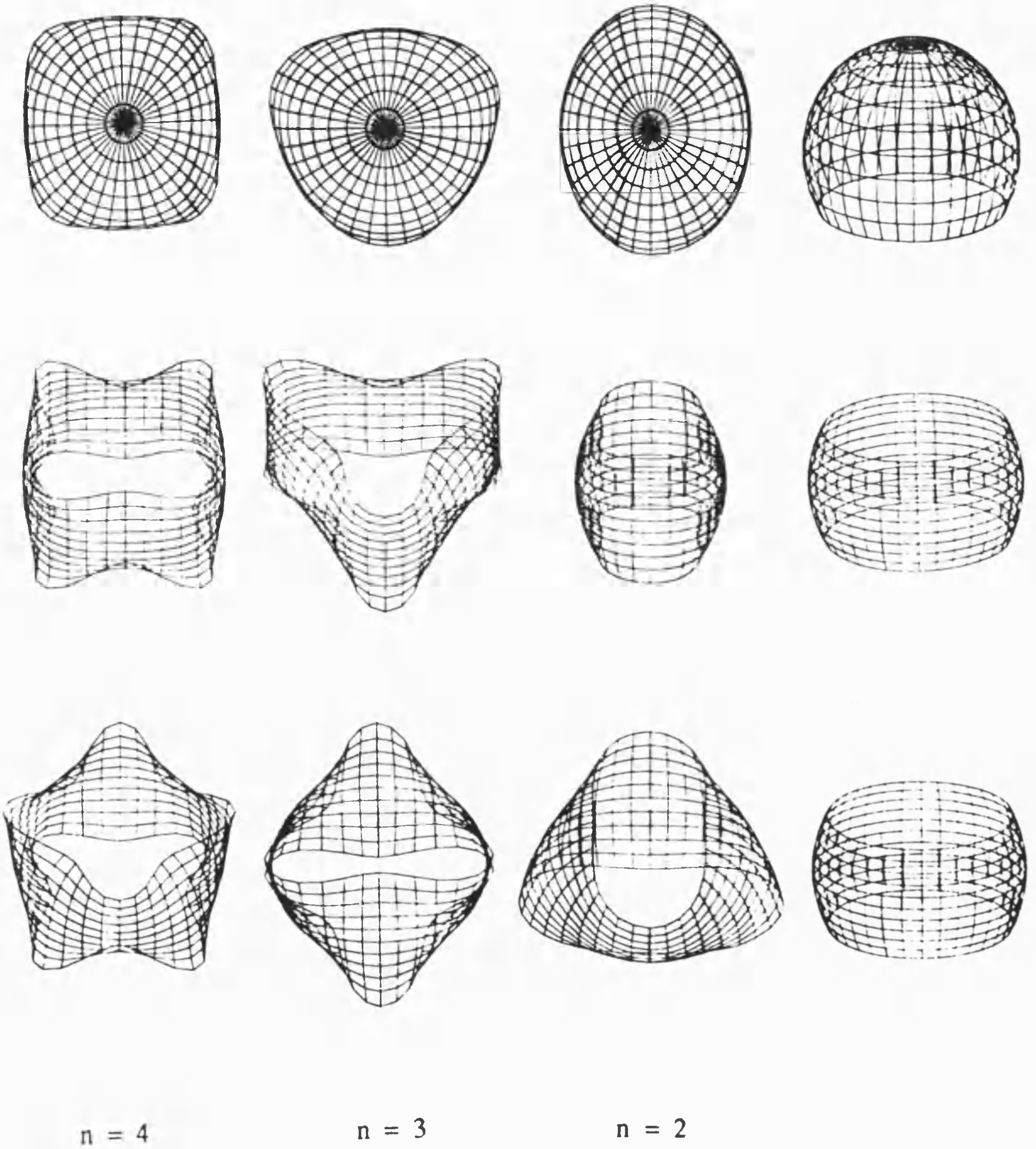


fig.(10_2.2) inextensional deformation of spherical shells for n≥2.

10_2.1 The boundary conditions of the Spherical shell

In practice the open spherical shell comes in two different forms according to whether the edge_line consists of one or two circles of latitude (spherical dome or cap and spherical belt respectively). Both cases are considered in the present analysis, due to their importance in construction and industry.

In order to examine the effect of adding stiffening beams to the boundary of a spherical shell we require the rates of change of curvature and twist at the boundary. The boundary will be assumed to lie on a line of latitude, ϕ = constant.

From (10_1.25), we have

$$\Omega^2|_2 = -(\varepsilon^{12})^2\Big\{ \Big[\Omega|_{22}\, b_{11} + \Omega|_2\, b_{11}|_2 - \Omega|_{12}\, b_{12} - \Omega|_1\, b_{12}|_2\Big]\frac{1}{K} - \frac{\big(\Omega|_2\, b_{11} - \Omega|_1 b_{12}\big)}{K^2}\, K|_2\Big\} \tag{10_2.10}$$

$$\Omega^2|_1 = -(\varepsilon^{12})^2\Big\{ \Big[\Omega|_{21}\, b_{11} + \Omega|_2\, b_{11}|_1 - \Omega|_{11}\, b_{12} - \Omega|_1\, b_{12}|_1\Big]\frac{1}{K} - \frac{\big(\Omega|_2\, b_{11} - \Omega|_1 b_{12}\big)}{K^2}\, K|_1\Big\}. \tag{10_2.11}$$

Upon using (10_1.17), (10_1.18) and the geometrical quantities derived above for the spherical shell, we write

$$\Omega^2|_1 = \frac{\cosh^2\phi}{a}\left[\Omega,_{21} + \Omega,_1 \tanh\phi\right]$$

$$\Omega^2|_2 = \frac{\cosh^2\phi}{a}\left[\Omega,_{22} + \Omega,_2 \tanh\phi\right].$$

Therefore, from (10_1.23)

$$\beta^{12} = \frac{\cosh^2\phi}{a^2}\left[\frac{\cosh^2\phi}{a}\left[\Omega,_{22} + \Omega,_2 \tanh\phi\right] - \Omega\, b_2^2\right]$$

$$\beta^{22} = -\frac{\cosh^2\phi}{a^2}\left[\frac{\cosh^2\phi}{a}\left[\Omega,_{21} + \Omega,_1 \tanh\phi\right]\right].$$

Thus, the rates of change of curvature and twist for the spherical shell along a line ϕ = constant are respectively

$$k_n^{\bullet} = -\frac{\cosh^2\phi}{a}\left[\Omega,_{21} + \Omega,_1 \tanh\phi\right]$$

$$\tau^{\bullet} = -\left[\frac{\cosh^2\phi}{a}\left[\Omega,_{22} + \Omega,_2 \tanh\phi\right] + \frac{\Omega}{a}\right]. \qquad (10\_2.12)$$

Performing the derivatives of the normal component, the above equations become

$$k_n^{\bullet} = \sum_{n=2}^{\infty} \frac{n\,(n^2-1)\cosh^2\phi}{a}\Bigg[\left[A_n \sinh n\phi + B_n \cosh n\phi\right]\sin n\vartheta - \left[C_n \sinh n\phi + D_n \cosh n\phi\right]\cos n\vartheta\Bigg]$$

$$\tau^{\bullet} = -\sum_{n=2}^{\infty} \frac{n\,(n^2-1)\cosh^2\phi}{a}\Bigg[\left[A_n \cosh n\phi + B_n \sinh n\phi\right]\cos n\vartheta + \left[C_n \cosh n\phi + D_n \sinh n\phi\right]\sin n\vartheta\Bigg] \qquad (10\_2.13)$$

For n equals, 0 and 1, both k_n^* and τ^* are equal to zero, as would be expected for a rigid body motion.

A_ Shell with one pole removed

If the north pole i.e $\phi = \infty$ is included, one has to put $A_n = -B_n$ and $C_n = -D_n$ in order that the solution should be finite, and thus

$$k_n^* = \sum_{n=2}^{\infty} \frac{n\,(n^2-1)\cosh^2\phi}{a} e^{-n\phi}\left[-A_n \sin n\vartheta + C_n \cos n\vartheta\right]$$

$$\tau^* = -\sum_{n=2}^{\infty} \frac{n\,(n^2-1)\cosh^2\phi}{a} e^{-n\phi}\left[A_n \cos n\vartheta + C_n \sin n\vartheta\right]$$

Thus it would appear that providing a beam sufficient to set $k_n^* = 0$ at the boundary is sufficient to prevent inextensional deformation. The provision of a lip as shown in fig.(10_2.3) has very much the same effect.

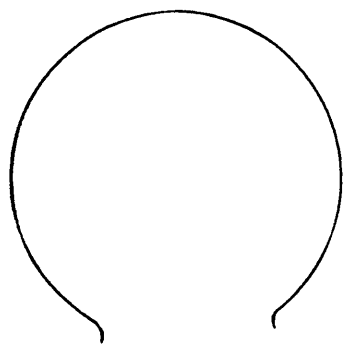


fig.(10_2.3) Spherical shell with a lip.

B_ Spherical belt $n \geq 2$

In this case we have two edge_line circles from which the two poles of the spherical shell have been removed. k^{*}_{n} and τ^{*} have to be finite between and on these edge_line circles. The fact that both poles have been removed means that the general solution:

$$k^{*}_{n} = \sum_{n=2}^{\infty} \frac{n\,(n^{2}-1)\,\cosh^{2}\phi}{a} \left[\left[A_{n} \sinh n\phi + B_{n} \cosh n\phi \right] \sin n\vartheta \right.$$

$$\left. - \left[C_{n} \sinh n\phi + D_{n} \cosh n\phi \right] \cos n\vartheta \right]$$

$$\tau^{*} = - \sum_{n=2}^{\infty} \frac{n\,(n^{2}-1)\,\cosh^{2}\phi}{a} \left[\left[A_{n} \cosh n\phi + B_{n} \sinh n\phi \right] \cos n\vartheta \right.$$

$$\left. + \left[C_{n} \cosh n\phi + D_{n} \sinh n\phi \right] \sin n\vartheta \right]$$

is finite throughout the belt. Thus inextensional deformation is possible unless suitable stiffening is provided along the edge_line circles.

First, let us provide beams at both edges sufficient to put $k^{*}_{n} = 0$. Then the following systems of equations is obtained for the two parallel circles bounding the shell at $\phi = \phi_{1}$ and $\phi = \phi_{2}$

$$\left[A_{n} \sinh n\phi_{1} + B_{n} \cosh n\phi_{1} \right] \sin n\vartheta = 0$$

$$\left[A_{n} \sinh n\phi_{2} + B_{n} \cosh n\phi_{2} \right] \sin n\vartheta = 0$$

and similarly

$$\left[C_n \sinh n\phi_1 + D_n \cosh n\phi_1\right]\cos n\vartheta = 0$$

$$\left[C_n \sinh n\phi_2 + D_n \cosh n\phi_2\right]\cos n\vartheta = 0.$$

In order that these systems of equations may have non_ zero solutions their determinants must vanish, i.e.

$$\Delta = \sinh n\phi_1 \cosh n\phi_2 - \sinh n\phi_2 \cosh n\phi_1 = \sinh n(\phi_1 - \phi_2).$$

$\Delta = 0$ only if $\phi_1 = \phi_2$, which is impossible. Therefore inextensional deformation is prevented by suitable beams or lips at the edge circles preventing the change in the normal curvature as shown in fig.(10_2.4).

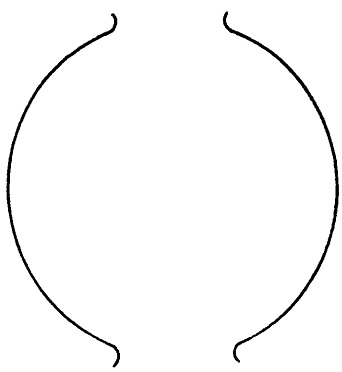


fig.(10_2.4) Spherical belt with lips

The second possibility is to fix completely one edge. This is equivalent to putting $\bar{k}_n = 0$ and $\bar{\tau}_n = 0$ at the same edge. which means that the following systems of equations are obtained

$$\left[A_n \sinh n\phi + B_n \cosh n\phi\right] \sin n\vartheta = 0$$

$$\left[A_n \cosh n\phi + B_n \sinh n\phi\right] \cos n\vartheta = 0$$

and similarly

$$\left[C_n \sinh n\phi + D_n \cosh n\phi\right] \cos n\vartheta = 0$$

$$\left[C_n \cosh n\phi + D_n \sinh n\phi\right] \sin n\vartheta = 0$$

where ϕ is the angle at the fixed edge.

Similarly, in order that these systems of equations may have non_zero solutions their determinants must vanish. However,

$$\Delta = \sinh^2 n\phi - \cosh^2 n\phi = -1$$

and therefore the determinant is never equal to zero, so that there is only the trivial solution $A_n = B_n = 0$ and $C_n = D_n = 0$. Thus providing the shell with a rigid base at only one edge (see fig.(10_2.5)) is sufficient to prevent inextensional deformation.

Note that fixing a membrane shell in position only controls the two components of displacement in the tangential direction. Displacement in the direction normal to the surface of the shell cannot be directly controlled due to the supposed absence of normal shear force.

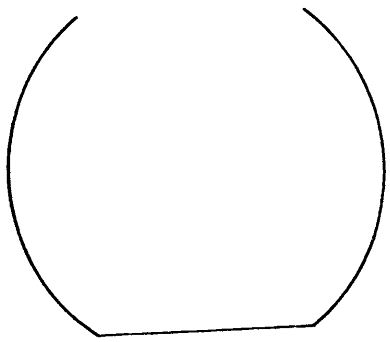


fig. (10_25) Spherical belt with rigid base

10_3 The Catenoid shell

The catenoid represents the category of surfaces that has a negative Gaussian curvature. In addition the catenoid is a minimal surface. According to O'Neill (1966), the meridians have the shape of a chain hanging under the influence of gravity, the parallel circles of the catenoid are given by

$$\left.\begin{aligned} r_0 &= a \cosh \phi \\ z &= a \phi \end{aligned}\right\}. \qquad (10\_3.1)$$

where a represents the radius at the waist of the shell, fig.(10_3.1)

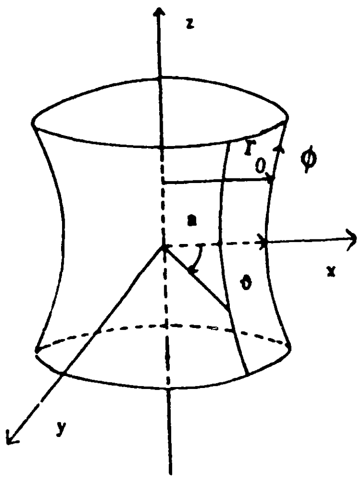


fig.(10_3.1) The catenoid shell

The position vector of a point on the surface of the catenoid is given by

$$\mathbf{r}(\phi,\vartheta) = a \cosh\phi \cos\vartheta\ \mathbf{i} + a \cosh\phi \sin\vartheta\ \mathbf{j} + a\ \phi\ \mathbf{k}. \qquad (10\_3.2)$$

The first two derivatives of r_0 and z are

$r_0' = a \sinh \phi \qquad\qquad r_0'' = a \cosh \phi$

$z' = a \qquad\qquad z'' = 0 .$

The base vectors of the catenoid are

$$\mathbf{a}_1 = a \cosh\phi \left[- \sin\vartheta\ \mathbf{i} + \cos\vartheta\ \mathbf{j}\right]$$
$$\mathbf{a}_2 = a\left[\sinh\phi \left[\ \cos\vartheta\ \mathbf{i} + \sin\vartheta\ \mathbf{j}\right] + \mathbf{k}\ \right]. \qquad (10\_3.3)$$

The metrics are

$$a_{11} = a^2\cosh^2\phi \qquad a_{22} = a^2\cosh^2\phi \qquad a_{12} = a_{21} = 0$$

$$a^{11} = \frac{1}{a^2\cosh^2\phi} \qquad a^{22} = \frac{1}{a^2\cosh^2\phi} \qquad a^{12} = a^{21} = 0.$$

Then, the contravariant base vectors are

$$\mathbf{a}^1 = \frac{1}{a\cosh\phi}\left[-\sin\vartheta\,\mathbf{i} + \cos\vartheta\,\mathbf{j}\right]$$

$$\mathbf{a}^2 = \frac{1}{a\cosh^2\phi}\left[\cos\vartheta\,\sinh\phi\,\mathbf{i} + \sin\vartheta\,\sinh\phi\,\mathbf{j} + \mathbf{k}\right].$$

The determinant A of the first fundamental form is

$$A = a_{11}a_{22} = a^4\cosh^4\phi.$$

The normal to the surface is

$$\mathbf{n} = \frac{1}{\cosh\phi}\left[\cos\vartheta\,\mathbf{i} + \sin\vartheta\,\mathbf{j} - \sinh\phi\,\mathbf{k}\right].$$

The derivatives of the base vectors are

$$\mathbf{a}_{1,1} = -\,a\cosh\phi\left[\cos\vartheta\,\mathbf{i} + \sin\vartheta\,\mathbf{j}\right]$$

$$\mathbf{a}_{2,2} = +\,a\cosh\phi\left[\cos\vartheta\,\mathbf{i} + \sin\vartheta\,\mathbf{j}\right]$$

$$\mathbf{a}_{1,2} = \mathbf{a}_{2,1} = a\sinh\phi\left[-\sin\vartheta\,\mathbf{i} + \cos\vartheta\,\mathbf{j}\right].$$

The curvature tensors are

$$b_{22} = -\,b_{11} = a \qquad b^1_1 = -\,b^2_2 = -\,\frac{1}{a\cosh^2\phi}$$

$$b_{12} = b_{21} = 0 \qquad b^1_2 = b^2_1 = 0\ .$$

The Christoffels for the catenoid are

$$\Gamma^1_{11} = 0 \qquad \Gamma^2_{22} = \tanh\phi$$

$$\Gamma^1_{21} = \tanh\phi \qquad \Gamma^2_{12} = 0$$

$$\Gamma^1_{22} = 0 \qquad \Gamma^2_{11} = -\tanh\phi$$

and the Gaussian and mean curvatures are

$$K = -\frac{1}{a^2\cosh^4\phi} \qquad H = 0 \quad \text{(minimal surface).}$$

We add the following special results

$$K|_1 = 0 \qquad K|_2 = \frac{4\tanh\phi}{a^2\cosh^4\phi}. \tag{10\_3.4}$$

Equation (10_2.5), with the use of (10_1.17), (10_3.4) and replacing the geometrical quantities of the catenoid, becomes

$$\Omega,_{11} - \Omega,_{22} - 2\,\Omega,_2 \tanh\phi = 0. \tag{10\_3.5}$$

Examination of the coefficients, shows that equation (10_3.5) is of a hyperbolic type. Then, if we write the solution in the following manner

$$\Omega = \sum_{n=0}^{\infty} \left[f_n(\phi) \cos n\vartheta + g_n(\phi) \sin n\vartheta \right] \tag{10\_3.6}$$

where $f_n(\phi)$ and $g_n(\phi)$ are function of ϕ only, we get

$$\Omega,_{11} = -n^2\Omega.$$

Then the periodicity of the solution reduces the problem to an ordinary differential equation, performing the derivatives of (10_3.6) and substituting in (10_3.5), we write

$$f_n''(\phi) + 2\, f_n'(\phi)\, \tanh\phi + n^2\, f_n(\phi) = 0$$

$$g_n''(\phi) + 2\, g_n'(\phi)\, \tanh\phi + n^2\, g_n(\phi) = 0.$$

This system of equation has the following solution for $f_n(\phi)$ and $g_n(\phi)$

$$f_n(\phi) = \frac{1}{\cosh\phi}\left[A_n \cos\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\} + B_n \sin\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}\right]$$

$$g_n(\phi) = \frac{1}{\cosh\phi}\left[C_n \cos\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\} + D_n \sin\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}\right].$$

(10_3.7)

Again A_n, B_n, C_n, D_n are constants and C_0 and D_0 can be taken equal to zero due to $\sin n\vartheta$ in (10_3.6)

Substituting (10_3.7) into (10_3.6), the normal component of the angular velocity vector becomes

$$\Omega = \frac{1}{\cosh\phi}\sum_{n=0}^{\infty}\left[\left[A_n \cos\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\} + B_n \sin\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}\right]\cos n\vartheta\right.$$

$$\left. + \left[C_n \cos\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\} + D_n \sin\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}\right]\sin n\vartheta\right].$$

(10_3.8)

Using equation (10_1.20), and substituting the corresponding geometrical quantities, we write the tangential components of the angular velocity vector for the catenoid in the following manner

$$\Omega^1 = \frac{n}{a\ \cosh\phi} \sum_{n=0}^{\infty} \Bigg[- \left[A_n \cos\left\{ \left[\sqrt{(n^2-1)} \right] \phi \right\} + B_n \sin \left\{ \left[\sqrt{(n^2-1)} \right] \phi \right\} \right] \sin n\vartheta$$

$$+ \left[C_n \cos \left\{ \left[\sqrt{(n^2-1)} \right] \phi \right\} + D_n \sin \left\{ \left[\sqrt{(n^2-1)} \right] \phi \right\} \right] \cos n\vartheta \Bigg]$$

$$\Omega^2 = - \frac{1}{a\ \cosh\phi} \sum_{n=0}^{\infty} \Bigg[\Bigg[- \left[A_n \sin\left\{ \left[\sqrt{(n^2-1)} \right] \phi \right\} - B_n \cos\left\{ \left[\sqrt{(n^2-1)} \right] \phi \right\} \right] \cos n\vartheta$$

$$- \left[C_n \sin \left\{ \left[\sqrt{(n^2-1)} \right] \phi \right\} - D_n \cos \left\{ \left[\sqrt{(n^2-1)} \right] \phi \right\} \right] \sin n\vartheta \Bigg] \sqrt{(n^2-1)}$$

$$- \frac{\sinh\phi}{\cosh\phi} \Bigg[\left[A_n \cos\left\{ \left[\sqrt{(n^2-1)} \right] \phi \right\} + B_n \sin \left\{ \left[\sqrt{(n^2-1)} \right] \phi \right\} \right] \cos n\vartheta$$

$$+ \left[C_n \cos \left\{ \left[\sqrt{(n^2-1)} \right] \phi \right\} + D_n \sin \left\{ \left[\sqrt{(n^2-1)} \right] \phi \right\} \right] \sin n\vartheta \Bigg] \Bigg] .$$

(10_3.9)

The velocity gradient for the catenoid is

$$v_{,1} = \sqrt{A} \left(- \Omega^2 n + \Omega\ a^2 \right) .$$

Substituting the values of the angular velocities for the catenoid, using its base vectors and integrating, we end up with the following expression

$$\mathbf{v} = a\sum_{n=2}^{\infty} \frac{-1}{\sqrt{(n^2-1)}}\left[n\left\{\left[A_n \sin\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\} - B_n \cos\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}\right]\sin n\vartheta\right.\right.$$

$$\left.\left. - \left[C_n \sin\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\} - D_n \cos\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}\right]\cos n\vartheta\right\}\left(\cos\vartheta\mathbf{i} + \sin\vartheta\mathbf{j}\right)$$

$$+\left\{\left[A_n \sin\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\} - B_n \cos\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}\right]\cos n\vartheta\right.$$

$$\left.\left. + \left[C_n \sin\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\} - D_n \cos\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}\right]\sin n\vartheta\right\}\left(-\sin\vartheta\mathbf{i} + \cos\vartheta\mathbf{j}\right)\right]$$

$$+a\sum_{n=2}^{\infty} \frac{1}{n}\left[\left\{\cosh\phi\left[A_n \cos\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\} + B_n \sin\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}\right]\right.\right.$$

$$\left. + \sqrt{(n^2-1)}\sinh\phi\left[A_n \sin\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\} - B_n \cos\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}\right]\right\}\sin n\vartheta$$

$$- \left\{\cosh\phi\left[C_n \cos\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\} + D_n \sin\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}\right]\right.$$

$$\left.\left. + \sqrt{(n^2-1)}\sinh\phi\left[C_n \sin\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\} - D_n \cos\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}\right]\right\}\cos n\vartheta\right]\mathbf{k} + \mathbf{c}. \qquad (10\_3.10)$$

$\mathbf{c}$ is a vector which corresponds to rigid body velocity.

Again, the velocity corresponding to the values of $n = 0$ and $n = 1$, with C_0 and D_0 both equal to zero, is

$$v = - a \left[A_0 \sinh\phi - i\, B_0 \cosh\phi\right]\left[-\sin\vartheta\, \mathbf{i} + \cos\vartheta\, \mathbf{j}\right] + a\, A_0 \vartheta\, \mathbf{k}$$

where $i = \sqrt{-1}$, and $n = 1$ gives

$$v=-a\left[\left[A_1\phi - \frac{B_1}{\sqrt{(n^2-1)}}\right]\mathbf{j} - \left[C_1\phi - \frac{D_1}{\sqrt{(n^2-1)}}\right]\mathbf{i} - \cosh\phi \left[A_1 \sin\vartheta - C_1 \cos\vartheta\right] \mathbf{k}\right]$$

First of all, B_0 has to be imaginary so that iB_0 becomes real. The constant A_0 in this case is of great importance, it produces a displacement in the **k** direction which varies linearly with the coordinate ϑ. This is only possible if a meridian from the catenoid is missing, i.e the catenoid is slit along the generator where then, its application eventually gives a new surface known as the helicoid, fig.(10_3.2).

However as we are concerned with inextensional deformation in which the first fundamental form of the surface is preserved, these two surfaces are identical from the point of view of intrinsic properties. Their different shapes are manifestations of their different second fundamental forms, see Lord & Wilson (1984). The present phenomenon is known in differential geometry as the local isometry, see also Do Carmo (1976).

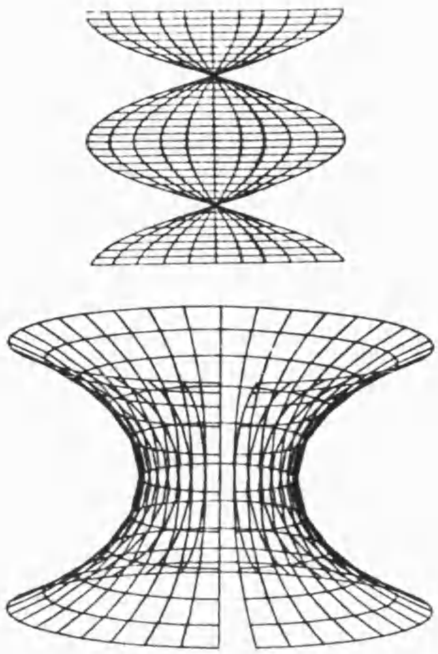


fig.(10_3.2) *The passage from slit catenoid to a skew helicoid*

The equations

$x = - \sinh\phi \sin\vartheta$, $y = \sinh\phi \cos\vartheta$, $z = \vartheta$

are the parametric equations for the position vector of helicoid surface, which is also a minimal surface. According to Do Carmo (1976), if **x** and **y** are two differentiable functions which satisfy Cauchy_Riemann equations, then they are seen to be harmonic and will be called harmonic conjugate. If **x** and **y** are minimal surfaces, which is the case of the helicoid and the catenoid, then they are called conjugate minimal surfaces.

Also, the surface

$$Z = \cos A \ \mathbf{x} + \sin A \ \mathbf{y}$$

is a minimal surface for all $A \in R$. If we put $\cos A = t$ then, we can write the general position vector for both surfaces as follows

$$x = \sqrt{(1 - t^2)}\cos\vartheta\cosh\phi - t\sin\vartheta\sinh\phi$$

$$y = \sqrt{(1 - t^2)}\sin\vartheta\cosh\phi + t\cos\vartheta\sinh\phi$$

$$z = \sqrt{(1 - t^2)}\,\phi + t\,\vartheta$$

with $t = 0$ for the catenoid , $t = 1$ skew helicoid.

However, as we are concerned with complete catenoid, both B_0 and A_0 have to be set equal to zero. The coefficients A_1 and C_1 produce rigid body angular velocities, and B_1, D_1 must tend to zero as n tends to 1. Thus, for the inextensional modes of deformation, we will consider only values of n which are ≥ 2.

The components of the velocity vector are

$$v^1 = \sum_{n=0}^{\infty} \frac{1}{\sqrt{(n^2-1)}\cosh\phi}\left\{\left[-A_n\sin\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}+B_n\cos\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}\right]\right.$$

$$\left.\cos n\vartheta + \left[-C_n\sin\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}+D_n\cos\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}\right]\sin n\vartheta\right\}$$

$$v^2 = \sum_{n=0}^{\infty} \frac{-\sinh\phi}{n\sqrt{(n^2-1)}\cosh^2\phi}\left\{\left[A_n\sin\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}-B_n\cos\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}\right]\right.$$

$$\sin n\vartheta - \left[C_n \sin\left\{\left[\surd(n^2-1)\right] \phi\right\} - D_n \cos\left\{\left[\surd(n^2-1)\right] \phi\right\}\right]\Bigg\}\cos n\vartheta\Bigg]\Bigg\}$$

$$+ \frac{1}{n \cosh\phi} \left\{\left[A_n \cos\left\{\left[\surd(n^2-1)\right] \phi\right\} + B_n \sin\left\{\left[\surd(n^2-1)\right] \phi\right\} \right]\sin n\vartheta\right.$$

$$\left. - \left[C_n \cos\left\{\left[\surd(n^2-1)\right] \phi\right\} + D_n \sin\left\{\left[\surd(n^2-1)\right] \phi\right\}\right]\cos n\vartheta\right\}$$

$$v^3 = \sum_{n=0}^{\infty} \frac{-\quad a}{\surd(n^2-1)\ \cosh\phi} \left(n^2\cosh^2\phi - \sinh^2\phi\right)$$

$$\times \left\{\left[A_n \sin\left\{\left[\surd(n^2-1)\right] \phi\right\} - B_n \cos\left\{\left[\surd(n^2-1)\right] \phi\right\} \right]\sin n\vartheta\right.$$

$$\left. - \left[C_n \sin\left\{\left[\surd(n^2-1)\right] \phi\right\} - D_n \cos\left\{\left[\surd(n^2-1)\right] \phi\right\}\right]\cos n\vartheta\right\}$$

$$- \frac{a\sinh\phi}{n} \left\{\left[A_n \cos\left\{\left[\surd(n^2-1)\right] \phi\right\} + B_n \sin\left\{\left[\surd(n^2-1)\right] \phi\right\} \right]\sin n\vartheta\right.$$

$$\left. - \left[C_n \cos\left\{\left[\surd(n^2-1)\right] \phi\right\} + D_n \sin\left\{\left[\surd(n^2-1)\right] \phi\right\}\right]\cos n\vartheta\right\}.$$

The corresponding graphical inextensional deformations for $n \geq 2$ are shown in fig.(10_3.3)

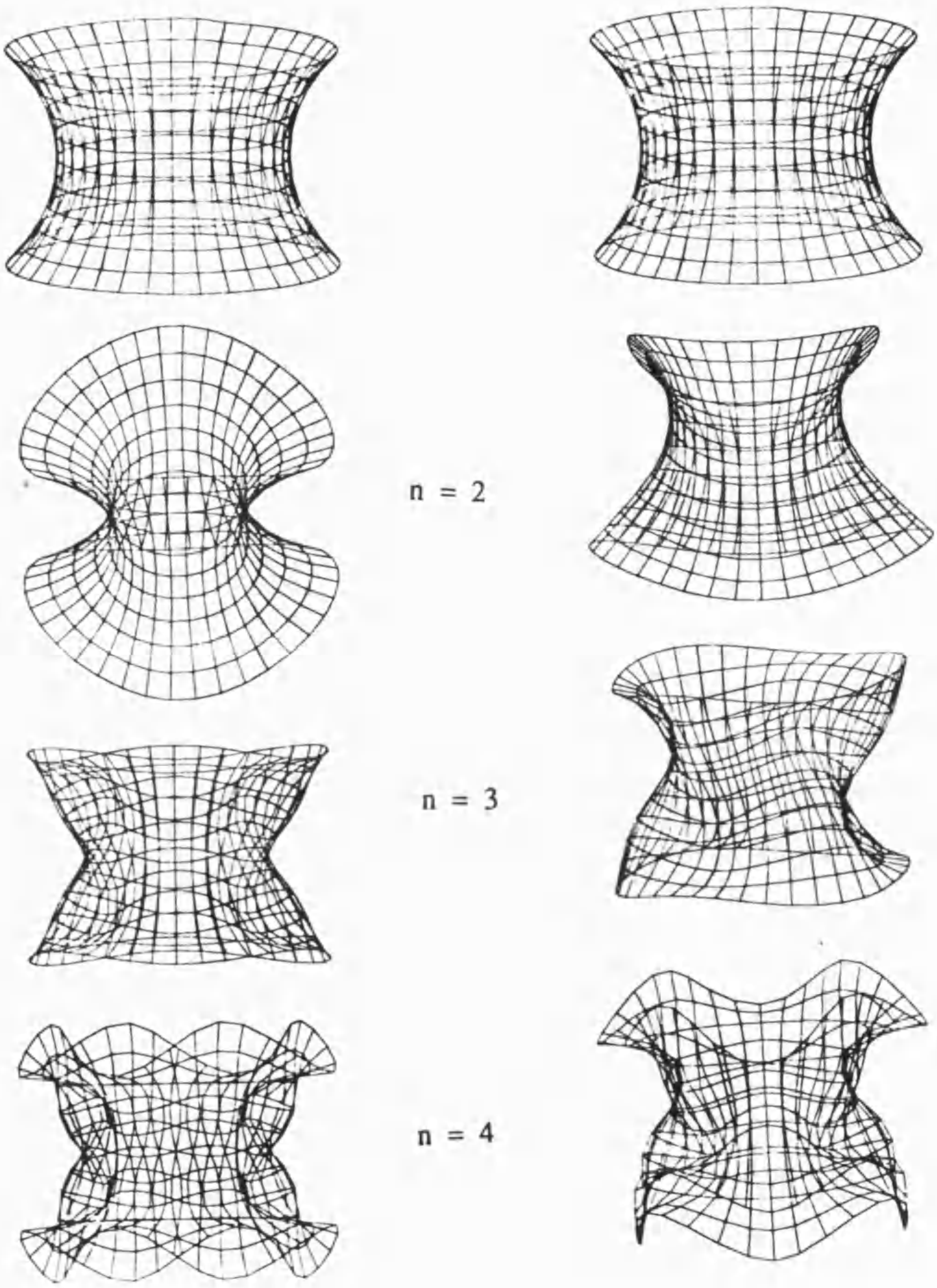


fig (10_3.3) *The inextensional modes of deformation of the catenoid for n≥2.*

10_3.1 The boundary conditions of the catenoid

Let us start with deriving the expressions of the rates of change of twist and normal curvature. Then from (10_1.21) and use of (10_2.10), (10_2.11), (10_1.17), (10_1.18) and using the geometrical quantities of the catenoid

$$\Omega^2|_2 = -\frac{1}{a}\left[\Omega_{,22} + \Omega_{,2}\tanh\phi\right]$$

$$\Omega^2|_1 = -\frac{1}{a}\left[\Omega_{,12} + \Omega_{,1}\tanh\phi\right]$$

and

$$k_n^{\bullet} = \frac{1}{a}\left[\Omega_{,21} + \tanh\phi\ \Omega_{,1}\right]$$

$$\tau^{\bullet} = \frac{1}{a}\left[\Omega_{,22} + \tanh\phi\ \Omega_{,2} + \frac{\Omega}{\cosh^2\phi}\right].$$

Performing the derivatives for the normal component of the angular velocity of the catenoid, considering only values of n which are ≥ 2, we write

$$k_n^{\bullet} = \frac{n\sqrt{(n^2-1)}}{a\cosh\phi}\sum_{n=2}^{\infty}\left[\left[A_n\sin\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\} - B_n\cos\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}\right]\sin n\vartheta\right.$$

$$\left. - \left[C_n\sin\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\} - D_n\cos\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}\right]\cos n\vartheta\right]$$

$$\tau^{*} = -\frac{1}{a}\sum_{n=2}^{\infty}\Bigg[\frac{\sqrt{(n^2-1)}\ \sinh\phi}{\cosh^2\phi}$$

$$\times\Bigg[-\left[A_n \sin\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\} - B_n \cos\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}\right]\cos n\vartheta$$

$$-\left[C_n \sin\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\} - D_n \cos\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}\right]\sin n\vartheta\Bigg]$$

$$+\frac{\sqrt{(n^2-1)}}{\cosh\phi}\Bigg[\left[A_n \cos\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\} + B_n \sin\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}\right]\cos n\vartheta$$

$$+\left[C_n \cos\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\} + D_n \sin\left\{\left[\sqrt{(n^2-1)}\right]\phi\right\}\right]\sin n\vartheta\Bigg]\Bigg].$$

(10_3.11)

The catenoid shell is bounded by two parallel circles at $\phi = \phi_1$ and $\phi = \phi_2$, the above expressions for the rate of changes of the normal curvature and twist are finite at the boundary edges. We will look at the problem of providing the appropriate boundary support to prevent inextensional deformation.

First, consider the case where we provide the shell with a stiffening beam at both edges to prevent changes in normal curvature along the edges. This produces the following systems of equations

$$\left[A_n \sin\left\{\left[\sqrt{(n^2-1)}\right]\ \phi_1\right\} - B_n \cos\left\{\left[\sqrt{(n^2-1)}\right]\ \phi_1\right\}\right] \sin n\vartheta = 0$$

$$\left[A_n \sin\left\{\left[\sqrt{(n^2-1)}\right]\ \phi_2\right\} - B_n \cos\left\{\left[\sqrt{(n^2-1)}\right]\ \phi_2\right\}\right] \sin n\vartheta = 0$$

and also

$$\left[\ C_n\ \sin\ \left\{\left[\sqrt{(n^2-1)}\right]\ \phi_1\right\} - D_n\ \cos\ \left\{\left[\sqrt{(n^2-1)}\right]\ \phi_1\right\}\ \right] \cos n\vartheta = 0$$

$$\left[\ C_n\ \sin\ \left\{\left[\sqrt{(n^2-1)}\right]\ \phi_2\right\} - D_n\ \cos\ \left\{\left[\sqrt{(n^2-1)}\right]\ \phi_2\right\}\ \right] \cos n\vartheta = 0.$$

In order that the above systems may have non_zero solution, their determinants Δ must vanish, then

$$\Delta = -\ \sin\left\{\left[\sqrt{(n^2-1)}\right]\ \phi_1\right\} \cos\left\{\left[\sqrt{(n^2-1)}\right]\phi_2\right\}$$

$$+\sin\left\{\left[\sqrt{(n^2-1)}\right]\ \phi_2\right\}\cos\left\{\left[\sqrt{(n^2-1)}\right]\phi_1\right\} = \sin\left\{\left[\sqrt{(n^2-1)}\right]\left[\phi_2 + \phi_1\right]\right\}.$$

The determinants vanish if

$$\left[\phi_2 + \phi_1\right] = \frac{m\ \pi}{\sqrt{(n^2-1)}},$$

where m and $n \geq 2$ are any integers.

Thus, the above formulated boundary conditions do not prevent inextensional deformation of the catenoid, unlike the spherical belt.

Consider now, the case where we provide the catenoid with rigid fixing of one edge, this corresponds to eliminating completely the rate of changes of the normal curvature and twist, i.e., $\dot{k}_n = 0$ and $\dot{\tau} = 0$. This leave us with the following system of equations

$$\left[A_n \sin\left\{\left[\sqrt{(n^2-1)}\right] \phi\right\} - B_n \cos\left\{\left[\sqrt{(n^2-1)}\right] \phi\right\}\right] = 0$$

$$\left[A_n \cos\left\{\left[\sqrt{(n^2-1)}\right] \phi\right\} + B_n \sin\left\{\left[\sqrt{(n^2-1)}\right] \phi\right\}\right] = 0$$

similarly, a system of equations is obtained for C_n and D_n. Thus, in order for these systems to have a non_zero solution their determinants must vanish, and we write

$$\Delta = \sin^2\left[\sqrt{(n^2-1)}\right] \phi + \cos^2\left[\sqrt{(n^2-1)}\right] \phi = 1.$$

The determinant is always distinct from zero and then the systems have only the trivial solutions $A_n = B_n = 0$ and $C_n = D_n = 0$. Thus two conditions at one edge of the catenoid remove completely the mechanism and allow for a membrane shell, see fig.(10_3.4).

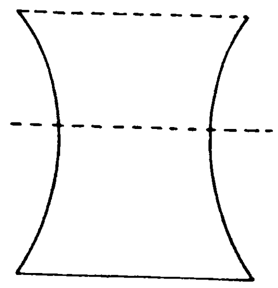


fig.(10_3.4) Catenoid shell completely fixed at one edge

10_4 The cylindrical shell

The cylinder in shell theory has a special treatment, due to its unique geometry. As our classification of shells depends on the geometry of the middle surface, we mention here that the cylinder represents the category of surfaces of vanishing Gaussian curvature. The parallel circles on the middle surface of the cylinder are given by the following special equations

$$\left.\begin{aligned} r_0 &= a \\ z &= a\,\phi \end{aligned}\right\} \qquad (10\_4.1)$$

where a is the radius of the cylinder, fig.(10_4.1)

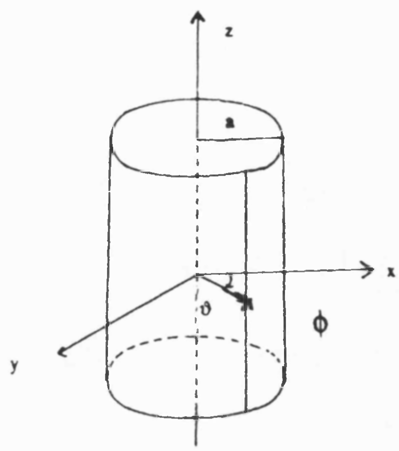


fig.(10_4.1) The cylindrical Shell

The position vector of a point on the surface of the cylinder is given by

$$\mathbf{r}(\phi,\vartheta) = a\cos\vartheta\ \mathbf{i} + a\sin\vartheta\ \mathbf{j} + a\ \phi\ \mathbf{k}. \tag{10\_4.2}$$

The first two derivatives of r_0 and z with respect to ϕ are

$$r_0' = 0 \qquad r_0'' = 0$$

$$z' = a \qquad z'' = 0.$$

The base vectors are obtained by differentiating $\mathbf{r}$ with respect to the two variables ϑ and ϕ, then

$$\mathbf{a}_1 = a \left[- \sin\vartheta \; \mathbf{i} + \cos\vartheta \; \mathbf{j} \right]$$

$$\mathbf{a}_2 = a \; \mathbf{k} \; . \qquad (10\_4.3)$$

The metrics are

$$a_{11} = a^2 \qquad a_{22} = a^2 \qquad a_{12} = a_{21} = 0$$

$$a^{11} = \frac{1}{a^2} \qquad a^{22} = \frac{1}{a^2} \qquad a^{12} = a^{21} = 0.$$

Then, the contravariant base vectors are

$$\mathbf{a}^1 = - \frac{1}{a} \left[- \sin\vartheta \; \mathbf{i} + \cos\vartheta \; \mathbf{j} \right]$$

$$\mathbf{a}^2 = \frac{1}{a} \; \mathbf{k} \; .$$

The determinant A of the first fundamental form is

$$A = a_{11} a_{22} = a^4.$$

The normal to the surface is

$$\mathbf{n} = \cos\vartheta \; \mathbf{i} + \sin\vartheta \; \mathbf{j}.$$

The derivatives of the base vectors are

$$\mathbf{a}_{1,1} = - \; a \left[\cos\vartheta \; \mathbf{i} + \sin\vartheta \; \mathbf{j} \right]$$

$$\mathbf{a}_{1,2} = \mathbf{a}_{2,1} = 0 \qquad \mathbf{a}_{2,2} = 0 \; .$$

The curvature tensors are

$$b_{11} = -\,a \qquad\qquad b^1_1 = -\,\frac{1}{a}$$

$$b_{12} = b_{21} = b^2_2 = 0 \qquad\qquad b^2_2 = b^1_2 = b^2_1 = 0.$$

The Christoffels for the cylinder are all equal to zero, so that the covariant derivatives are the same as the partial derivatives.

The Gaussian and mean curvatures are

$$K = 0 \text{ (developable surface)} \qquad\qquad H = -\,\frac{1}{2\,a}.$$

As we mentioned before, equation (8_3.19) can not be applied for the inextensional deformation of the cylinder for the simple raison of the vanishing of K, instead we use (8_3.14). It produces two tangential equation according to α being 1 and 2

$$\Omega^{\beta}\, b_{\beta\alpha} + \Omega|_{\alpha} = 0 \quad \begin{cases} -\,a\ \Omega^1 + \Omega|_1 = 0 \\ \\ \Omega|_2 = 0 \end{cases} \tag{10_4.4}$$

upon substituting the geometrical quantities for the cylinder into the general equation. Taking advantage again of the periodicity of the cylinder, the normal component of the angular velocity will have the following solution

$$\Omega = \sum_{n=0}^{\infty} \left[A_n \cos n\vartheta + B_n \sin n\vartheta \right] \tag{10_4.5}$$

where A_n and B_n are the constants of integration and $B_0 = 0$. Then, from the first equation of (10_4.4), we write

$$\Omega^1 = \frac{1}{a} \sum_{n=0}^{\infty} n \left[- A_n \sin n\vartheta + B_n \cos n\vartheta \right] . \qquad (10\_4.6)$$

From equation (8_3.13), we deduce the second tangential component of the angular velocity

$$\Omega^2 = \frac{(n^2-1)}{a} \sum_{n=0}^{\infty} \left\{ \left[A_n \cos n\vartheta + B_n \sin n\vartheta \right] \phi + \left[C_n \cos n\vartheta + D_n \sin n\vartheta \right] \right\}$$

(10_4.7)

with C_n and D_n are constants and $D_0 = 0$.

From (8_3.3), integrating with respect to ϑ, the corresponding vector velocity of the cylinder is

$$\mathbf{v} = a \sum_{n=0}^{\infty} n \left\{ - \left[A_n \sin n\vartheta - B_n \cos n\vartheta \right] \phi - C_n \sin n\vartheta + D_n \cos n\vartheta \right\} x \left(\cos\vartheta\ \mathbf{i} + \sin\vartheta\ \mathbf{j} \right)$$

$$-a \sum_{n=0}^{\infty} \left\{ \left[A_n \cos n\vartheta + B_n \sin n\vartheta \right] \phi + C_n \cos n\vartheta + D_n \sin n\vartheta \right\} x \left(-\sin\vartheta\ \mathbf{i} + \cos\vartheta\ \mathbf{j} \right)$$

$$+a \sum_{n=0}^{\infty} \frac{1}{n} \left[A_n \sin n\vartheta - B_n \cos n\vartheta \right] \mathbf{k} + \mathbf{c} \qquad (10\_4.8)$$

where **c** corresponds to rigid body velocity.

Here also the contributions to **v** from n = 0 and n = 1 are

$$-a \left[A_0\ \phi + C_0 \right]\ x\ \left[-\sin\vartheta\ \mathbf{i} + \cos\vartheta\ \mathbf{j} \right] + A_0 \vartheta\ \mathbf{k}$$

where B_0 must be put equal to zero as n tends to zero.

$$a\left\{-\left[\ A_1\phi + C_1\ \right]\mathbf{j}+\left[\ B_1\ \phi + D_1\right]\mathbf{i}+\left[A_1\sin\vartheta\ -\ B_1\cos\vartheta\right]\mathbf{k}\right\}.$$

Again these displacements represent rigid body velocities and angular velocities and will not contribute to the inextensional deformation.

A_0 produces an angular velocity about the axes X and Y, and also a velocity similar to that found in the catenoid, which is possible only if the cylinder is slit along the meridian, fig.(10_4.2). A similar case has been discussed by Calladine (1983).

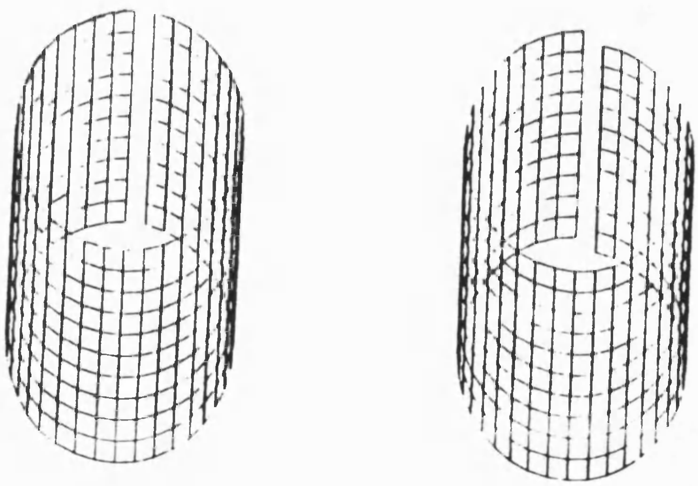


fig.(10_4.2) The effect of A_0 along the axis of the shell

A_0 must be set equal to zero. C_0 produce a rigid body velocity in X and Y directions. A_1, B_1 produce rigid body angular velocities about X and Y axes and C_1, D_1 rigid body velocities along the same axes. Therefore, in the discussion of the boundary conditions necessary to prevent inextensional deformation, we will consider only values for n which are ≥ 2.

The components of the velocity vector are

$$v^1 = -\sum_{n=0}^{\infty}\left\{\left[A_n \cos n\vartheta + B_n \sin n\vartheta\right]\phi + C_n \cos n\vartheta + D_n \sin n\vartheta\right\}$$

$$v^2 = \sum_{n=0}^{\infty}\frac{1}{n}\left[A_n \sin n\vartheta - B_n \cos n\vartheta\right]$$

$$v^3 = -\sum_{n=0}^{\infty} a\, n\left\{\left[A_n \sin n\vartheta - B_n \cos n\vartheta\right]\phi + C_n \sin n\vartheta - D_n \cos n\vartheta\right\}.$$

The inextensional modes of deformation corresponding to $n \geq 2$ are shown in fig.(10_4.3)

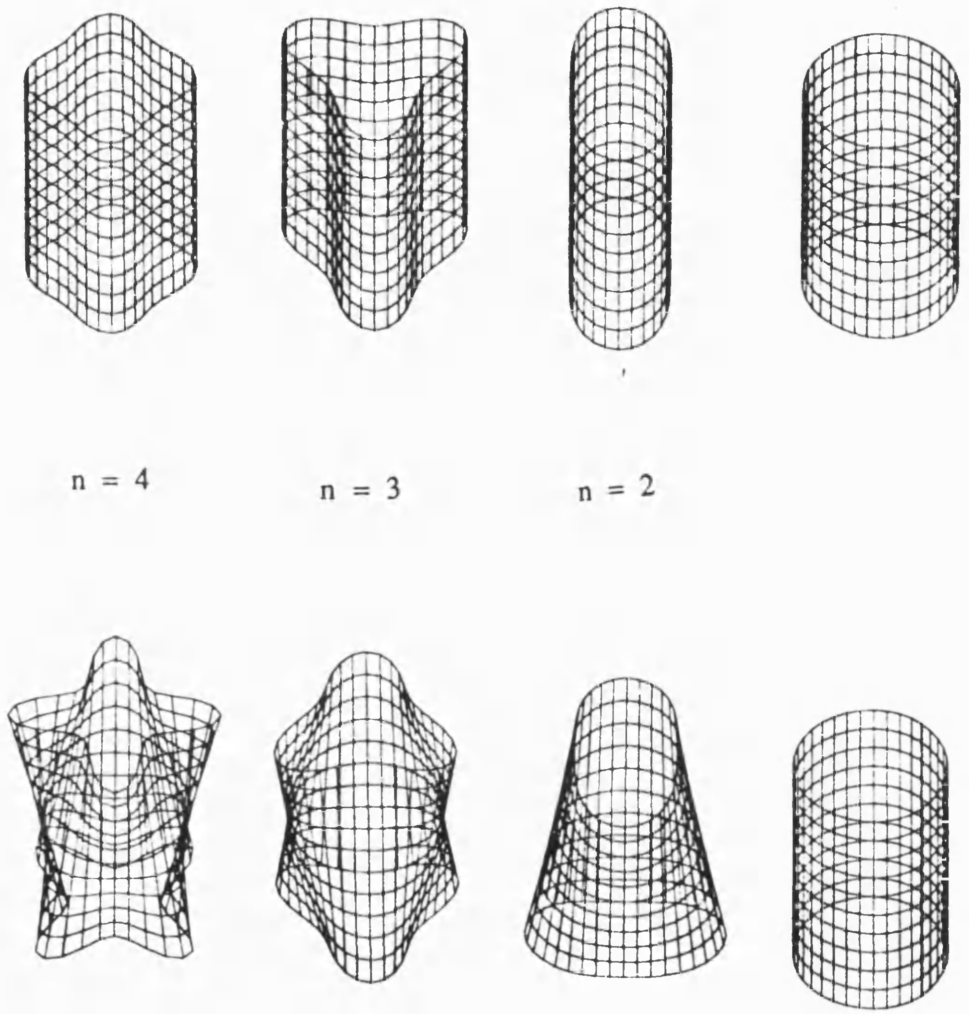


fig.(10_4.3) Inextensional modes of deformation

for cylinder, n≥ 2

10_4.1 The boundary conditions of the cylinder

To derive the rates of change of twist and normal curvature for the cylinder, we avoid the previous approach used for the catenoid and the spherical shell because of the appearance of terms containing the Gaussian curvature. Instead we use the second order tensor $c_{\alpha\beta}$, then differentiating (3_3.31) with respect to time and using (8_4.15), we have

$$k_n^{\bullet} = - \frac{c_{\alpha\beta}\, d\vartheta^{\alpha}\, d\vartheta^{\beta}}{a_{\eta\gamma}\, d\vartheta^{\eta}\, d\vartheta^{\gamma}} \quad \text{and} \quad \tau^{\bullet} = \frac{c_{\alpha\beta}\, a^{\gamma\alpha}\varepsilon_{\lambda\gamma}\, d\vartheta^{\beta}\, d\vartheta^{\lambda}}{a_{\eta\gamma}\, d\vartheta^{\eta}\, d\vartheta^{\gamma}}.$$

The boundary lines are given by the equation $\vartheta^2=$ const., therefore $d\vartheta^2 = 0$, and the above equations become

$$k_n^{\bullet} = - \frac{c_{11}}{a_{11}} \qquad \text{and} \qquad \tau^{\bullet} = \frac{c_{21}\, a^{22}\varepsilon_{12}}{a_{11}}.$$

Using (8_4.1), we write

$$k_n^{\bullet} = \frac{\Omega,_1 \varepsilon_{12}.a^2}{a_{11}} \qquad \text{and} \qquad \tau^{\bullet} = - \frac{\Omega,_2\, a^{22}(\varepsilon_{12})^2.a^2}{a_{11}}.$$

Substituting the corresponding geometrical quantities of the cylinder and performing the different derivatives of the angular velocity vector, we write finally

$$k^*_n = \sum_{n=2}^{\infty} \frac{n\,(\,n^2-1\,)}{a} \left\{ \phi\left[-A_n \sin n\vartheta + B_n \cos n\vartheta\right] - C_n \sin n\vartheta + D_n \cos n\vartheta \right\}$$

$$\tau^* = -\sum_{n=2}^{\infty} \frac{(\,n^2-1\,)}{a} \left[A_n \cos n\vartheta + B_n \sin n\vartheta\right]. \qquad (10\_4.9)$$

First, we note that the twist and also the symmetrical part of k^*_n are independent of the meridian coordinates ϕ. This is due to the characteristic curves that constitute the meridians of the cylinder. Quantities that have this property are prescribed only once on the surface.

From the above two equations we see that by providing one of the edges of the cylinder with a rigid base that prevent both k^*_n and τ^* , fig.(10_4.4), we get the following system of equations, $\phi = 0$

$$k^*_n = 0 \qquad -B_n \sin n\vartheta + D_n \cos n\vartheta = 0$$

$$\tau^* = 0 \qquad A_n \cos n\vartheta + C_n \sin n\vartheta = 0.$$

This system has only the trivial solution $B_n = D_n = 0$ and $A_n = C_n = 0$. Therefore the rigid base at only one edge remove the mechanism.

Also consider the case where a beam capable of putting $k^*_n = 0$ at each end is provided at the two edges, fig.(10_4.4) of the cylinder, then we write the following system of equations

$$\phi = 0 \qquad -B_n \sin n\vartheta + D_n \cos n\vartheta = 0$$

$$\phi = 1 \qquad -(A_n + B_n)\sin n\vartheta + (C_n + D_n)\cos n\vartheta = 0.$$

Again this way of supporting the cylinder remove the mechanism.

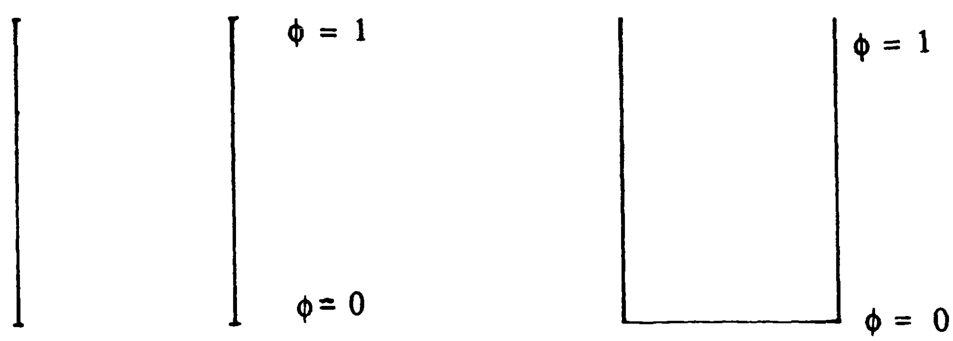


fig.(10_4.4). cylinders constituting a structure

10_5 Infinitesimal bending of non_convex shells

In the results in chapter 9 concerning the rigidity of compact surfaces, the proofs required the convexity of the surfaces. This rises the question as to what happens when the convexity condition is relaxed. Apparently, there are no general results or theorems concerning this matter. However, particular cases of surfaces that are non_convex can be investigated.

In the existing theory we mention a particular case given by

Flugge (1967). It consists of two spherical shells connected at the waist circle as shown in fig.(10_5.1).

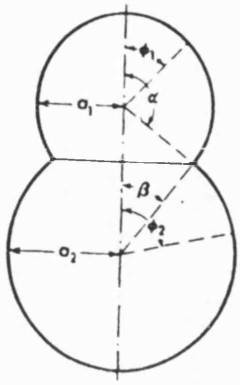


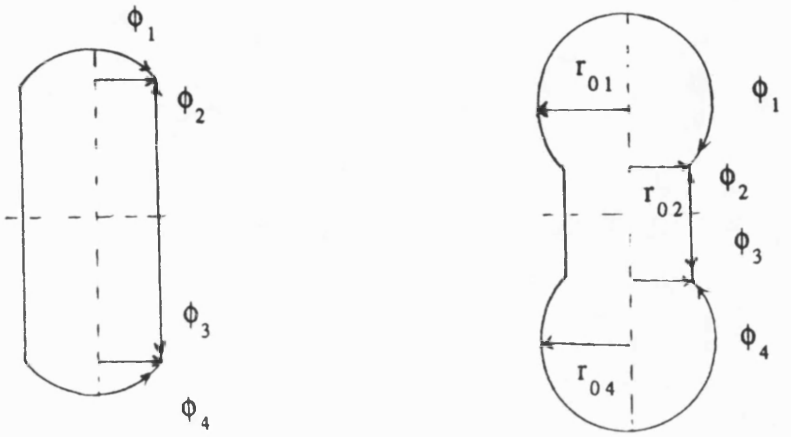
fig. (10_5.1) Spherical shells connected at the waist circle

Flugge (1967), showed that for

$$n = \frac{\sin \frac{\alpha + \beta}{2}}{\sin \frac{\alpha - \beta}{2}} \qquad (10\_5.1)$$

(where n is an integer representing the n^{th} harmonic of Fourier series for the displacement's solutions), the numerator is always positive for $\alpha<\beta$ (convex shell) and this can never yield a positive n. However, for $\alpha>\beta$ (non_convex shell) as shown in fig.(10_5.1), the possibility for the structure being non_rigid is only when equation (10_5.1) is satisfied i.e. the equation gives an integer. This example shows the possibility of constructing infinitesimally rigid non_convex shell structures.

Also let us examine the case of the cylinder closed with spherical ends where the possibility of non_convexity can be obtained by varying the meridian coordinate of the spherical caps as shown in fig.(10_5.2).



A: convex shell *B: non_convex shell*

fig.(10_5.2) cylindrical shell closed with spherical caps

At the parallel circles where the spherical caps and the cylinder meet, the displacements of the sphere and cylinder must be the same. From the velocity vectors for both the cylinder and the spherical shell, three equations Vx, Vy and Vz can be obtained between them, where only two are independent. These are:

A)_ <u>Upper spherical cap</u>

For a finite solution where the north pole ϕ tends to ,+ ∞, is included we must put $A_n = - B_n$ and $C_n = - D_n$, then

$$V_x = \frac{a_1}{\cosh\phi_1}\left\{A_n e^{-n\phi_1}\left(n\sin n\vartheta\ \cos\vartheta - \cos n\vartheta\ \sin\vartheta\right) - \right.$$

$$\left. C_n e^{-n\phi_1}\left(n\cos n\vartheta\ \cos\vartheta + \sin n\vartheta\ \sin\vartheta\right)\right\}$$

$$V_z = a_1\left\{A_n e^{-n\phi_1}\left(n\ \tanh\ \phi_1 + 1\right)\sin n\vartheta - C_n e^{-n\phi_1}\left(n\ \tanh\ \phi_1 + 1\right)\cos n\vartheta\right\}.$$

B)_ Lower spherical cap

When the south pole $\phi = -\infty$ is included, a finite solution is obtained by putting $A_n = B_n$ and $C_n = D_n$, then we write

$$V_x = -\frac{a_4}{\cosh\phi_4}\left\{A_n e^{n\phi_4}\left(n\ \sin\ n\vartheta\ \cos\vartheta - \cos\ n\vartheta\ \sin\vartheta\right) - \right.$$

$$\left. C_n e^{n\phi_4}\left(n\ \cos\ n\vartheta\ \cos\vartheta + \sin\ n\vartheta\ \sin\vartheta\right)\right\}$$

$$V_z = -a_4\left\{A_n e^{n\phi_4}\left(n\tanh\ \phi_4 - 1\right)\sin n\vartheta - C_n e^{n\phi_4}\left(n\tanh\ \phi_4 - 1\right)\cos n\vartheta\right\}.$$

C)_ The cylinder

$$V_x = a\left\{A_n\left(-n\sin n\vartheta\ \cos\vartheta + \cos n\vartheta\ \sin\vartheta\right) + B_n\left(n\cos n\vartheta\ \cos\vartheta + \sin n\vartheta\ \sin\vartheta\right)\right\}\phi$$

$$a\left\{-C_n\left(n\sin n\vartheta\ \cos\vartheta - \cos n\vartheta\ \sin\vartheta\right) + D_n\left(n\cos n\vartheta\ \cos\vartheta + \sin n\vartheta\ \sin\vartheta\right)\right\}$$

$$V_z = \frac{a}{n}\left(A_n\sin n\vartheta - B_n\cos n\vartheta \right).$$

For the cylinder the radius a is constant and ϕ becomes ϕ_1 and ϕ_2 for the upper and lower connections respectively.

We check the rigidity of surfaces such those shown in fig.(10_5.2) by setting the following equalities at each connection,

$Vx_{\,sph.} = Vx_{\,cyl.}$, $Vz_{\,sph.} = Vz_{\,cyl.}$

For a symmetrical structure as shown in fig.(10_5.2), either a symmetrical or an anti_symmetrical deformation can be taken, then

$$-\frac{a_1}{\cosh\phi_1} C_{nsph.}\, e^{-n\phi_1} n = a_2 D_{ncyl.}\, n$$

$$a_1 C_{nsph.}\, e^{-n\phi_1}\left(n \tanh \phi_1 + 1\right) = a_2 D_{ncyl.} / n$$

$$\frac{a_4}{\cosh\phi_4} A_{nsph.}\, e^{n\phi_4} n = a_2 D_{ncyl.}\, n$$

$$a_4 A_{nsph.}\, e^{n\phi_4}\left(n \tanh \phi_4 - 1\right) = - a_2 D_{ncyl.} / n.$$

The following system of equations is obtained

$$\frac{a_1}{\cosh\phi_1} C_n e^{-n\phi_1} n + \frac{a_4}{\cosh\phi_4} A_n e^{n\phi_4} n = 0$$

$$a_1 C_n e^{-n\phi_1}\left(n \tanh \phi_1 + 1\right) + a_4 A_n e^{n\phi_4}\left(n \tanh \phi_4 - 1\right) = 0.$$

In order that the system may have non_zero solution, the determinant Δ must vanish, i.e.

$$\frac{1}{\cosh\phi_1}\left(n \tanh \phi_4 - 1\right) - \frac{1}{\cosh\phi_4}\left(n \tanh \phi_1 + 1\right) = 0.$$

$$n = \frac{\cosh\phi_4 + \cosh\phi_1}{\sinh\phi_4 - \sinh\phi_1} = \coth \frac{1}{2}(\phi_4 - \phi_1) \tag{10_5.2}$$

The above relations are constructed on the basis that n is positive for a finite solution on the sphere. Equation (10_5.2) for n, is negative if $(\phi_4 - \phi_1) < 0$, i.e. $\phi_1 > \phi_4$. This condition shows that all convex surfaces of the form shown in fig.(10_5.2) A, are infinitesimally rigid. There are also possibilities where the condition $\phi_1 > \phi_4$ is satisfied giving non_convex surfaces and yet n is negative. This shows that non_convex surfaces can be made infinitesimally rigid.

On the other hand if $(\phi_4 - \phi_1) > 0$ i.e., $\phi_4 > \phi_1$, the shell is non_convex and n is positive, the mechanism appears only one n is an integer.

A particular case can be of importance if we assume that the tangent to the parallel circles where the different structures meet is continuous. this gives

$$r_{o\,sph.} = r_{o\,cyl.}\ ,\ r'_{o\,sph.} = r'_{o\,cyl.}$$

$$\frac{a_1}{\cosh\phi_1} = a_2 \quad , \cosh\phi_1 = \frac{a_1}{a_2}$$

$$-\frac{a_1 \sinh\phi}{\cosh^2\phi_1} = 0\ , \qquad \sinh\phi_1 = 0, \qquad \tanh\phi_1 = 0$$

Similarly at the lower connection

$$\cosh\phi_4 = \frac{a_4}{a_2} \quad , \sinh\phi_4 = 0 \ , \ \tanh\phi_4 = 0.$$

Then, the determinant Δ will be

$$\Delta = \frac{a_2}{a_1}\left(-1\right) - \frac{a_2}{a_4}\left(+1\right) = 0$$

a_2 is never equal 0, then $a_1 = -a_4$ which is impossible for a real structure. This confirms the result found in chapter 9 in which convex shells that have parabolic points are infinitesimally rigid.

However, the non_convexity in the above examples is not obtained from parts of the structure that have a negative Gaussian curvature, instead it is obtained from joining parts of the structure which have positive Gaussian curvature where the connection represents a reentrant angle. For the former case, consider the torus which is a non_convex surface of revolution of variable Gaussian curvature. It has been proved by Minagawa & Rado (1952) that the torus is an infinitesimally rigid surface. However, Gol'denveizer (1961) showed that when the torus is subject to statically equilibrated system of loading, fig.(10_5.3), a membrane state of stress turns to be impossible.

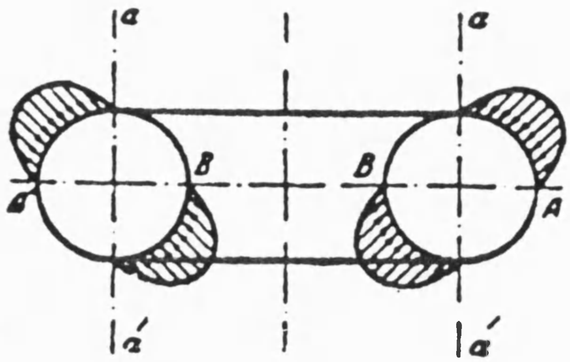


fig.(10_5.3) Torus under statically equilibrated loads

This can be shown by removing part B (cutting the torus by a cylinder along a-a′) and replace it by the action of membrane forces applied to the part A along the section, the equilibrium becomes impossible because the membrane forces do not have any component along the vertical axis. Kuznetsov (1989), explained the peculiar behaviour of the torus by the presence of asymptotic lines that facilitates the non_smooth infinitesimal bending with the very inception of loading.

The catenoid was proved to be infinitesimally rigid if at one of the edges a rigid base is provided. In the following example we will investigate what will happen if we close the catenoid with a spherical cap, fig.(10_5.4). In the same manner as for the cylinder, using the velocity vector of the catenoid we have

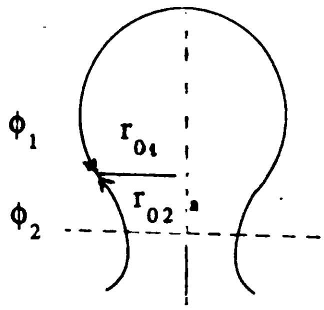


fig.(10_5.4) Catenoid closed at one edge by spherical cap.

$$V_x=-\frac{a}{\sqrt{(n^2-1)}}\left\{\left(A_n\sin\sqrt{(n^2-1)}\phi-B_n\cos\sqrt{(n^2-1)}\phi\right)\left(n\sin n\vartheta\cos\vartheta-\cos n\vartheta\sin\vartheta\right)\right.$$

$$\left.-\left(C_n\sin\sqrt{(n^2-1)}\phi-D_n\cos\sqrt{(n^2-1)}\phi\right)\left(n\cos n\vartheta\cos\vartheta+\sin n\vartheta\sin\vartheta\right)\right\}$$

$$V_z=\frac{a}{n}\left\{\left(A_n\left(\cosh\phi\ \cos\sqrt{(n^2-1)}\phi\ +\ \sqrt{(n^2-1)}\ \sinh\phi\ \sin\sqrt{(n^2-1)}\phi\right)\ +\right.\right.$$

$$\left.+\ B_n\left(\cosh\phi\ \sin\sqrt{(n^2-1)}\phi\ -\ \sqrt{(n^2-1)}\ \sinh\phi\ \cos\sqrt{(n^2-1)}\phi\right)\right)\sin\vartheta$$

$$-\ \left(C_n\left(\cosh\phi\ \cos\sqrt{(n^2-1)}\phi\ +\ \sqrt{(n^2-1)}\ \sinh\phi\ \sin\sqrt{(n^2-1)}\phi\right)\ +\right.$$

$$\left.\left.+\ D_n\left(\cosh\phi\ \sin\sqrt{(n^2-1)}\phi\ -\ \sqrt{(n^2-1)}\ \sinh\phi\ \cos\sqrt{(n^2-1)}\phi\right)\right)\cos n\vartheta\right\}.$$

By equating Vx and Vz for both the catenoid and the spherical cap at the parallel circle were they meet and combining the symmetrical and the anti_symmetrical solutions for this particular case, we get the following system of equations

$$-\frac{a_1}{\cosh\phi_1} C_{nsph.} e^{-n\phi_1} - \frac{a_2}{\surd(n^2-1)}\left(C_n \sin\surd(n^2-1)\phi_2 - D_n \cos\surd(n^2-1)\phi_2\right) = 0$$

$$-a_1 C_{nsph.} e^{-n\phi_1}\left(n\tanh\phi_1+1\right)+$$

$$\frac{a_2}{n}\left(C_n\left(\cosh\phi_2\cos\surd(n^2-1)\phi_2+\surd(n^2-1)\ \sinh\phi_2\sin\surd(n^2-1)\phi_2\right)+\right.$$

$$\left.D_n\left(\cosh\phi_2\sin\surd(n^2-1)\phi_2-\ \surd(n^2-1)\ \sinh\phi_2\cos\surd(n^2-1)\phi_2\right)\right) = 0.$$

In this particular case we do not need to check the determinant, since the system permits a non_trivial solutions. The structure is then, a mechanism.

Consider now the catenoid closed with spherical caps at both edges, fig.(10_5.5). Then the following system of equations is obtained

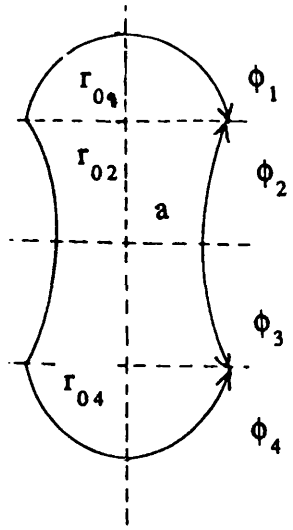


fig.(10_5.5) Catenoid closed with spherical caps.

$$-\frac{a_1}{\cosh\phi_1} C_{nsph.} e^{-n\phi 1} = \frac{a_2}{\sqrt{(n^2-1)}} C_{ncat.} \sin\sqrt{(n^2-1)}\phi_2$$

$$-a_1 C_{nsph.} e^{-n\phi 1}\left(n\tanh\phi_1 + 1\right) =$$

$$\text{X}$$

$$-\frac{a_2}{n} C_{ncat}\left(\cosh\phi_2 \cos\sqrt{(n^2-1)}\phi_2 + \sqrt{(n^2-1)}\ \sinh\phi_2 \sin\sqrt{(n^2-1)}\phi_2\right)$$

$$\frac{a_4}{\cosh\phi_4} C_{nsph1} e^{n\phi 4} = \frac{a_2}{\sqrt{(n^2-1)}} C_{ncat} \sin\sqrt{(n^2-1)}\phi_3$$

$$a_4 C_{nsph1} e^{n\phi 4}\left(n\tanh\phi_4 - 1\right) =$$

$$\text{Y}$$

$$-\frac{a_2}{n} C_{ncat}\left(\cosh\phi_3 \cos\sqrt{(n^2-1)}\phi_3 + \sqrt{(n^2-1)}\ \sinh\phi_3 \sin\sqrt{(n^2-1)}\phi_3\right) = 0.$$

Simplifying the above relations, we get

$$-\frac{a_1}{\cosh\phi_1 \sin\sqrt{(n^2-1)}\phi_2} C_{nsph.} e^{-n\phi 1} - \frac{a_4}{\cosh\phi_4 \sin\sqrt{(n^2-1)}\phi_3} C_{nsph1} e^{n\phi 4} = 0$$

$$\frac{-a_1 \, \bar{e}^{n\phi 1}\left(n \tanh\phi_1 + 1\right) C_{nsph.}}{X} - \frac{a_4 \, e^{n\phi 1}\left(n \tanh\phi_4 - 1\right) C_{nsph1}}{Y} = 0.$$

In order that the system may have non zero solution the determinant Δ must vanish i.e.,

$$\Delta = \left(n\sinh\phi_4 - \cosh\phi_4\right)\sin\sqrt{(n^2-1)}\phi_3 X - \left(n\sinh\phi_1 - \cosh\phi_1\right)\sin\sqrt{(n^2-1)}\phi_2 Y = 0.$$

In a similar way to previous examples, at the points where the different structures meet, we require the continuity of the tangent to the meridians.

$$\cosh\phi_1 = \frac{a_1}{a_2 \cosh\phi_2} \qquad , \ \sinh\phi_1 = -\frac{a_1 \sinh\phi_2}{a_2 \cosh^2\phi_2}$$

$$\cosh\phi_4 = \frac{a_4}{a_2 \cosh\phi_3} \qquad , \ \sinh\phi_4 = -\frac{a_4 \sinh\phi_3}{a_2 \cosh^2\phi_3}.$$

If we make the assumption that $\phi_2 = -\phi_3$ on the catenoid, the determinant becomes

$$\Delta = \frac{n \sinh\phi_2}{\cosh^2\phi_2}\left(\frac{a_1}{a_2} - \frac{a_4}{a_2}\right) + \frac{1}{\cosh\phi_2}\left(\frac{a_4}{a_2} - \frac{a_1}{a_2}\right) = 0.$$

Finally, we get

$$n = \coth\phi_2 \qquad (10\_5.3)$$

The coordinate ϕ_2 in this case is always positive, the structure becomes a mechanism whenever n constitutes an integer.

10_6 Conclusion

In this chapter we have shown that particular examples of compact non_convex surfaces are mechanisms if their geometry satisfies a certain conditions. For example, the catenoid closed by two spherical caps is a mechanism if equation (10_5.3) is satisfied.

If such condition is not satisfied, then the structure is not a mechanism, but we would expect the structure to be inefficient since it will usually be "almost a mechanism".

It was shown that the catenoid closed with only one sphere is a mechanism and this is consistent with the general result in section (9_3.3) concerning floating surfaces with a hole.

The following table shows results for non_compact surfaces where the openings are reinforced by a beam or a rigid base.

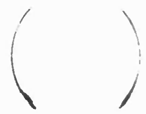
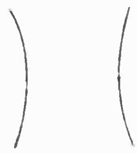
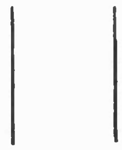
			
Beam prevents curvature change at each boundary	Structure	Mechanism	Structure
In plane displacement prevented at base	Structure	Structure	Structure

Table (4): The boundary supports for different structures

CHAPTER ELEVEN

CONCLUSIONS

11_1 Results and discussion

In the present work the problem of designing shells that work primarily by membrane action has been investigated. The aim of this work is to advance the understanding of the importance of boundary support for shell structures. The work should help engineers decide the form and support required for shells in civil engineering and mechanical engineering applications.

Shells that are thin enough to be represented by surfaces and thick enough to resist compressive stresses have also the property of being easily bent rather than stretched. On this basis, a combination of the statics of shells represented by membrane theory and the kinematics of shells represented by the inextensional deformation has produced a new parameter called the angular velocity vector. This parameter characterizes the different modes of deformation of the surface, i.e. the deformed element of the shell is fully defined by the components of the angular velocity vector.

By combining the equations of inextensional deformation, single partial differential equations have been obtained and used to find the boundary conditions necessary to prevent inextensional deformation.

Preventing inextensional deformation in shell structures means designing membrane shells which are infinitesimally rigid.

The rigidity of shells was found to be dependent on the geometry of structure and on the boundary supports.

In complete compact surfaces, application of the Cohn_Vossen theorem and its extended version due to Spivak (1979) has led to the conclusions

1)_ Strictly convex surfaces such as ovaloids are infinitesimally rigid.

2)_ Simply convex surfaces such as closed cylinder with spherical caps are also infinitesimally rigid.

3)_ Convex surfaces containing planar points constitute mechanisms.

It is also demonstrated in this thesis that

4)_ Floating surfaces with a hole are mechanisms.

5)_ Compact non_convex surfaces either containing regions of negative Gaussian curvature, or else composed from a combination of convex surfaces can be infinitesimally rigid under special circumstances.

The investigation of open surfaces required solutions to the single differential equations of the angular velocity, except for shells of parabolic surfaces where special procedure has been adopted. Particular distribution of the boundary supports to prevent inextensional deformation in the different cases of surface geometry has been found necessary:

1)_ For shells of negative Gaussian curvature with tow edges, one edge should be completely fixed with a rigid base, and the remote edge should be set free if infinitesimal bending is to be prevented.

2)_ For shells of zero Gaussian curvature, one rigid beam at each edge or complete fixing of one edge with a rigid base is found necessary for the structure to be infinitesimally rigid.

3)_ For shells of positive Gaussian curvature with one edge, one rigid beam is sufficient to prevent the mechanism.

4)_ For shells of positive Gaussian curvature with two edges, a complete fixing of one edge with rigid base or a rigid beam at each edge remove completely the mechanism.

When work on this thesis was started, the aim was to produce simple general rules which dictate whether a shell can work primarily by membrane action.

Rules have been discovered for certain shapes of shells, but there is still no overall set of rules for all shells. Thus there is scope for further work on this topic, but there is no obvious direction in which the work should proceed.

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